Complex Functions Or Conformal Maps

in 3D-XplorMath, a Visualization Program

Elementary Functions

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Elliptic Functions

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Complex Square $z \to z^2$ *

We use the parameter dependent mapping $z \mapsto aa(z - bb)^{ee} + cc$, with the default values $aa = 1$, $bb = 0$, $cc = 0$, and $ee = 2$. The default `Morph` joins z^2 to the identity, varying $ee \in [1, 2]$.

Look at the discussion in "About this Category" for what to look at, what to expect, and what to do.

Just as the appearance of the graph of a real-valued function $x \mapsto f(x)$ is dominated by the critical points of f, it is an important fact that so also, for a conformal map, $z \mapsto f(z)$, the overall appearance of an image grid is very much dominated by those points z where the derivative f' vanishes. Most obviously, near points a with $f'(a) = 0$ the grid meshes get very small and, as a consequence, the grid lines usually are strongly curved. If one looks more closely then one notices that the angle between curves through a is **not** the same as the angle between the image curves through $f(a)$ (recall: $f'(a) = 0$). We will find this general description applicable to many examples.

One should first look at the behaviour of the simple quadratic function $z \to z^2$ near $z = 0$, both in Cartesian and in Polar coordinates. One sees that a rectangle, which touches $z = 0$ from one side is folded around 0 with strongly curved parameter lines, and one also sees in Polar coordi-

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nates that the angle between rays from 0 gets **doubled**. The image grid in the Cartesian case consists of two families of orthogonal intersecting parabolas.

One should return to this prototype picture after one has seen others like $z \to z + 1/z$, $z \to z^2 + 2z$ and even the Elliptic functions and looked at the behaviour near their critical points.

The first examples to look at, (using Cartesian **and** Polar Grids) are $z \to z^2$, $z \to 1/z$, $z \to \sqrt{z}$, $z \to e^z$.

H.K.

The Complex Exponential Map $z \mapsto e^z$ *

Our example is $z \mapsto \exp(aa(z - bb)) + cc$, its parameters are set to $aa = 1$, $bb = 0$, $cc = 0$; aa gets morphed in $\mathbb{C}$. See the functions $z \mapsto z^2$, $z \mapsto 1/z$ and their ATOs first.

The complex exponential function $z \mapsto e^z$ is one of the most marvellous functions around. It shares with the real function $x \mapsto \exp(x)$ the differential equation $\exp' = \exp$ and the functional equation $\exp(z + w) = \exp(z) \cdot \exp(w)$. This latter identity implies that one can understand the complex Exponential in terms of real functions, for if we put $z = x + i \cdot y$ then we have

$$\exp(x + i \cdot y) = \exp(x) \cdot \exp(i \cdot y) =$$
$$\exp(x) \cdot \cos(y) + i \cdot \exp(x) \cdot \sin(y).$$

This says that a Cartesian Grid is mapped "conformally" (i.e., preserving angles) to a Polar Grid: the parallels to the real axis are mapped to radial lines, and segments of length 2π that are parallel to the imaginary axis are mapped to circles around 0. This function is therefore used to make, in the Action Menu, the Conformal Polar Grid. Observe how justified it is to describe the image grid as "made out of curved small squares".

If you have seen $z \mapsto e^z$ and $z \mapsto z + 1/z$ then now look at $z \mapsto \sin(z)$.

H.K.

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The Complex Map $z \mapsto 1/z$ *

The actual mapping for this example is $z \mapsto aa/(z-bb)+cc$, with the default values $aa = 1$, $bb = 0$, and $cc = 0$.

Look at the function $z \mapsto z^2$ and its ATO first.

The function $z \mapsto 1/z$ should be looked at both in Cartesian and Polar Grids. The default `Morph` varies $bb \in [0, 1]$.

Notice first:

1) The real axis, imaginary axis and unit circle are mapped into themselves,
2) the upper half plane and the lower half plane are interchanged, and
3) the inside of the unit circle and its outside are also interchanged.

This is best seen in the (default) Conformal Polar Grid. In the Cartesian Grid one should in particular observe that all straight parameter lines (in the domain) are mapped to circles (some exceptions, like the real axis, remain lines). The behaviour of these circles near zero can be looked at as an image of the behaviour of the standard Cartesian Grid near infinity. In fact **all** circles are mapped to circles or lines.

Examples to look at after this are

$z \mapsto (az + b)/(cz + d)$ and $z \mapsto (z + cc)/(1 + \bar{c}cz)$,

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both of which can be obtained from $z \to 1/z$ by composition with translations $z \to z+a$ or scaled rotations $z \to a \cdot z$. Therefore all of these so-called “Möbius transformations” map circles and lines to circles and lines.

H.K.

Complex Map $z \mapsto z + aa/z$ *

See the functions $z \to z^2$, $z \to 1/z$, $z \to z^2 + 2z$, $z \to e^z$ and their ATOs first. Use the default `Morph`, $aa \in [0, 1]$.

This function is best applied to a Conformal Polar Grid. The image of the outside of the unit circle is the same as the image of the inside of the unit circle, namely the full plane minus the segment $[-2, 2]$. The unit circle is mapped to this intervall, each interior point $w = 2 \cdot \cos(\phi) \in [-2, 2]$ appears twice as image point, namely of $z = \exp(\pm i\phi)$.

The default choice shows how the outside of the unit disk is mapped to the outside of the interval $[-2, 2]$. If we note that $f'(\pm 1) = 0$ then we understand this behaviour: the interior $180°$ angle at these critical points ± 1 of the outside domain is again **doubled** to become the angle of the image domain (outside $[-2, 2]$) at ± 2.

A domain circle $z_R(\phi) = R\exp(i\phi)$ is mapped to the image ellipse $(R+1/R)\cos(\phi) + i(R-1/R)\sin(\phi)$, and a domain radius $z_\phi(R) = R\exp(i\phi)$ is mapped to the Hyperbola $(R+1/R)\cos(\phi) + i(R-1/R)\sin(\phi)$. The image grid therefore consists of a *family of ellipses* that intersect orthogonally a *family of hyperbolae*, and all these Conic Sections (see the Plane Curves Category) are "confocal", i.e., they have the **same Focal Points**, namely at $+2$ and -2.

H.K.

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Complex Map $z \mapsto aa \cdot z^{ee} + ee \cdot z$ *

(Default: $z \to z^2 + 2z$)

Look at the functions $z \to z^2$, $z \to 1/z$ and their ATOs first. The default `Morph` varies $aa \in [0, 1.2]$ for $ee = 4$.

Of course, since $z^2 + 2z + 1 = (z+1)^2$, this function is not very different from the first example $z \to z^2$. But the change puts the critical point to -1 on the unit circle ($f'(-1) = 0$). Therefore, if one looks what this map does to a Polar Grid, one can study the behaviour near the critical point $z = -1$ with a different grid picture than in the first example. Circles outside the unit circle are mapped to Limaçons (Plane Curves Category) which wind around -1 twice. The unit circle is mapped to a Cardioid and one can see the interior angle of $180°$ of the unit circle at -1 mapped to the interior angle of $360°$ of the Cardioid at -1. Also one can see that a neigbourhood of -1 is strongly contracted by this function.
See the function $z \to z + 1/z$ next.

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Fractional Linear Maps*

or Möbius Transformations

$$z \mapsto (a \cdot z + b)/(c \cdot z + d)$$

See the functions $z \mapsto z^2$, $z \to 1/z$ and their ATOs first. The default `Morph` uses a conformal polar grid and varies $a \in [1, 2],\ c \in [0, 1]$.

These functions are called or fractional linear maps or **Möbius transformations**. They differ from the map $z \mapsto 1/z$ by composition with a translation $z \mapsto z + a$ or scaled rotations $z \mapsto a \cdot z$. As discussed for $z \mapsto 1/z$ they transform lines and circles to lines and circles.

The default special case is $z \mapsto (z - 1)/(z + 1)$. It is best understood in the (default) Conformal Polar Grid. Since it maps 0 to -1 and ∞ to $+1$, one can see the Polar coordinate centers moved from $0, \infty$ to $-1, +1$. This picture is the first step towards understanding the complex (or "Gaussian") plane plus the point at infinity as the "Riemann Sphere".

See also the other Möbius transformations from the Conformal Maps menu.

H.K.

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The Möbius Transformation*

$$z \to \frac{(z+cc)}{(1+\overline{cc}\cdot z)} \text{ of the unit disk.}$$

Look at the Möbius transformation $z \to \frac{(a\cdot z+b)}{(c\cdot z+d)}$ and its ATO first. The default `Morph` varies $cc \in [-0.9,\ 0.9]$.

This function maps the interior of the unit disk bijectively to itself, for every choice of cc with $|cc| < 1$. The behaviour outside of the unit disk is obtained by reflection in the unit circle, i.e., $z \to 1/\bar{z}$.

These maps have an interesting geometric interpretation: they are isometries for the "hyperbolic metric" on the unit disk. To understand this further, imagine that the unit disk is a map of this two-dimensional hyperbolic world and that the scale of this map is not a constant but equals $1/(1 - z\bar{z})$. This means that we do not obtain the length of a curve $t \to z(t)$ as in the Euclidean plane by the integral $\int |z'(t)|dt$—we have to take the scale into account and define its hyperbolic length by $\int |z'(t)|/(1 - |z(t)|^2)dt$. It is this hyperbolic length of curves that is left invariant by the "hyperbolic translations" $z \to (z + cc)/(1 + \overline{cc} \cdot z)$.

Locally the Pseudosphere (Category: Surfaces) has the same hyperbolic geometry.

H.K.

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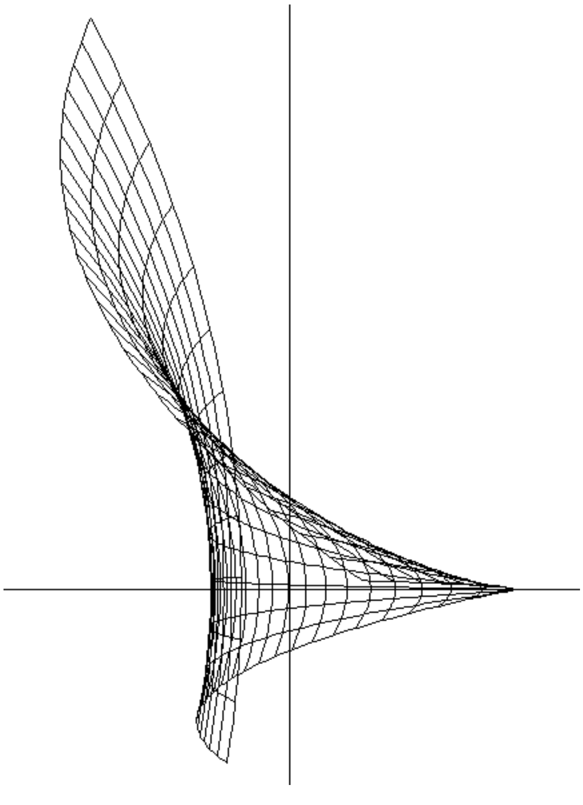
Nonconformal Complex Map $z \mapsto \mathrm{conj}(z) + aa \cdot z^2$ *

Look first at other functions and their ATOs, for example $z \to z^2$ and exp. The default `Morph` varies $aa \in [0, 1]$. The map $z \mapsto \mathrm{conj}(z) + aa \cdot z^2$ is a map from the complex plane to itself. The harmless looking "conj" is responsible for the fact that this map is not complex differentiable and therefore not a "conformal" map, that is, a map for which the angles between any two curves and their images are the same. It is clearly visible in the image that the squares of the domain grid are mapped to rectangles and even to parallelograms in the range.



The image also shows two "fold lines". We observe that interior points of the domain are mapped so that they lie on the boundary of the image. For a complex differentiable function this can never happen as is asserted by the *Open Mapping Theorem.* See the default morph.

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The Complex Map $z \mapsto \sqrt{z}$ *

See the functions $z \mapsto z^2$, $z \mapsto 1/z$ and their ATOs first. The map $z \mapsto \sqrt{z}$ should be looked at both in Cartesian and Polar Grids and in the default morph $z^{ee}, ee \in [\frac{1}{2}, 1]$.

Note that since this function is the inverse of $z \to z^2$, we expect to see related phenomena: circles around 0 go to circles around 0, radial lines from 0 go to radial lines from 0, but now with **half** the angle between them (since we look at the inverse map). A neigbourhood of 0 was very much contracted by $z \to z^2$, now we see the opposite, the distance of points from zero is increased very much (beyond any Lipschitz bound).
A more complicated aspect is the fact, since all $z \neq 0$ have two distinct square roots differing by a factor of -1, the function $z \mapsto \sqrt{z}$ is not really a well defined map until we make some choices.

The function $\sqrt{z}$ used by 3D-Filmstrip maps the upper half plane to the first quadrant, the (strict) lower half plane to the fourth quadrant, and the negative real axis to the positive imaginary axis—so there is no continuity from above to below the negative real axis (which is therefore called a "branch cut").

The Cartesian grid lines are mapped to two families of **hyperbolae** which intersect each other orthogonally.

H.K.

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The Complex Logarithm*

$$z \mapsto \log z$$

Look at the function $z \to e^z$ and its ATO first. The default `Morph` shows $a \cdot \log(z) + (1-a) \cdot z$ for $a \in [0, 1]$.

The complex Logarithm tries to be the inverse function of the complex Exponential. However, exp is $2\pi i$-periodic, so such an inverse can only exist as a multivalued function.

From the differential equation $\exp' = \exp$ follows that the derivative of the inverse is not multivalued and in fact very simple:

$$\log'(z) = 1/z.$$

Integration of the geometric series

$$\begin{aligned} 1/z &= 1/(1-(1-z)) = \textstyle\sum_k (1-z)^k \\ &= \big(\textstyle\sum_k -(1-z)^{k+1}/(k+1)\big)' \end{aligned}$$

gives the Taylor expansion around 1 of log. The so called "principal value" of the complex Logarithm is defined in the whole plane, but slit along the negative real axis, for example by integrating the derivative $\log'(z) = 1/z$ in that simply connected domain along any path starting at 1. Different values of $\log z$ differ by integer multiples of $2\pi i$, e.g. $i = \exp(\pi i/2)$ implies $\log i = \pi i/2 + 2\pi i \cdot \mathbb{Z}$.

H.K.

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The Complex Sine $z \to \sin(z)$ *

Look at the functions $z \to z^2$, $z \to 1/z$, $z \to z^2 + 2z$, $z \to e^z$ and their ATOs first. The default `Morph` varies the family $f_a(z) = a \cdot \sin(z) + (1 - a) \cdot z$ for $a \in [0, 1]$.

While the behaviour of the one-dimensional real functions $x \mapsto \exp(x)$ and $x \mapsto \sin(x)$ are quite dissimilar (exp is convex and positive, while sin is periodic and bounded), as complex functions they are very closely related:

$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i},$$

an identity that explains why the image grid under sin of the default Cartesian grid looks exactly like the image grid under $z \to z + 1/z$ applied to a Conformal Polar Grid outside the unit circle. For if we put $w(z) := \exp(iz)/i$, then $\sin(z) = (w(z) + 1/w(z)))/2$, and: recall that exp maps the standard Cartesian Grid to the Conformal Polar Grid around 0. The parameter curves in the image grid of sine are therefore the same orthogonal and confocal ellipses and hyperbolas as in the image of $z \mapsto z + 1/z$.

H.K.

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The Complex Tangent Hyperbolicus *

$$z \mapsto \tanh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}} = -i\; tan(i\,z)$$

See the functions $z \to z^2$, $z \to 1/z$, $z \mapsto (a{\cdot}z{+}b)/(c{\cdot}z{+}d)$, $z \to e^z$, $z \to \sin(z)$, and their ATOs first. The default `Morph` varies the parameter $a \in [0, 1]$ in the formula

$$f(z) := \tanh(a \cdot z/2) + (1 - a) \cdot z/2\ .$$

All real functions that have power series representations can be extended to be functions over the complex plane. Of course this includes all functions that have simple definitions in terms of the exponential map.
Maybe it comes as a surprise that the image net of the function tanh is the same as that of $(z-1)/(z+1)$. But if we abbreviate $w = \exp(2\,z)$ then we have, very similar to the case of the sine function, $\tanh(z) = (w-1)/(w+1)$. Now recall that we use a conformal polar grid in the domain to show $(z-1)/(z+1)$ and we use the standard cartesian grid in the domain to show tanh. The connection between the two domain grids is given by the complex exponential map.

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Symmetries Of Elliptic Functions*

[The approach below to elliptic functions follows that given in section 3 of "The Genus One Helicoid and the Minimal Surfaces that led to its Discovery", by David Hoffman, Hermann Karcher, and Fusheng Wei, published in Global Analysis and Modern Mathematics, Publish or Perish Press, 1993. For convenience, the full text of section 3 (without diagrams) has been made an appendix to the chapter on the Conformal Map Category in the documentation of 3D-XplorMath.]

An elliptic function is a doubly periodic meromorphic function, $F(z)$, on the complex plane $\mathbb{C}$. The subgroup $\mathbb{L}$ of $\mathbb{C}$ consisting of the periods of F (the period lattice) is isomorphic to the direct sum of two copies of $\mathbb{Z}$, so that the quotient, $T = \mathbb{C}/\mathbb{L}$, is a torus with a conformal structure, i.e., a Riemann surface of genus one. Since F is well-defined on $\mathbb{C}/\mathbb{L}$, we may equally well consider it as a meromorphic function on the Riemann surface T.

It is well-known that the conformal equivalence class of such a complex torus can be described by a single complex number. If we choose two generators for $\mathbb{L}$ then, without changing the conformal class of $\mathbb{C}/\mathbb{L}$, we can rotate and scale the lattice so that one generator is the complex num-

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ber 1, and the other, τ, then determines the conformal class of T. Moreover, τ_1 and τ_2 determine the same conformal class if and only if they are conjugate under $SL(2, Z)$.

The simplest elliptic functions are those defining a degree two map of T to the Riemann sphere. We will be concerned with four such functions, that we call JD, JE, JF, and WP. The first three are closely related to the classical Jacobi elliptic functions, but have normalizations that are better adapted to certain geometric purposes, and similarly WP is a version of the Weierstrass $\wp$-function, with a geometric normalization. Any of these four functions can be considered as the projection of a branched covering over the Riemann sphere with total space T, and as such it has four branch values, i.e., points of the Riemann sphere where the ramification index is two. For JD there is a complex number D such that these four branch values are $\{D, -D, 1/D, -1/D\}$. Similarly for JE and JF there are complex numbers E and F so that the branch values are $\{E, -E, 1/E, -1/E\}$ and $\{F, -F, 1/F, -1/F\}$ respectively, while for WP there is a complex number P such that the branch values are $\{P, -1/P, 0, \infty\}$. The cross-ratio, λ, of these branch values (in proper order) determines τ and likewise is determined by τ.

The branch values E, F, and P of JE, JF, and WP can be easily computed from the branch value D of JD (and hence from dd) using the following formulas:

$$E = (D-1)/(D+1), \quad F = -i(D-i)/(D+i),$$
$$P = i(D^2+1)/(D^2-1),$$

and we will use D as our preferred parameter for describing the conformal class of T. In 3D-XplorMath, D is related to the parameter dd (of the Set Parameter... dialog) by $D = \exp(dd)$, i.e., if $dd = a + ib$, the $D = \exp(a)\exp(ib)$. This is convenient, since if D lies on the unit circle (i.e., if dd is imaginary) then the torus is rectilinear, while if D has equal real and imaginary parts (i.e., if $b = \pi/4$) then the torus is rhombic. (The square torus being both rectilinear and rhombic, corresponds to $dd = i \cdot \pi/4$).

To completely specify an elliptic function in 3D-XplorMath, choose one of JD, JE, JF, or WP from the Conformal Map menu, and specify dd in the Set Parameter... dialog. (Choosing Elliptic Function from the Conformal map menu will give the default choices of JD and a square torus.)

When elliptic functions where first constructed by Jacobi and by Weierstrass these authors assumed that the lattice of the torus was given. On the other hand, in Algebraic Geometry, tori appeared as elliptic curves. In this representation the branch values of functions on the torus are given with the equation, while an integration of a holomorphic form (unique up to a multiplicative constant) is required to find the lattice. Therefore the relation between the period quotient τ (or rather its $SL(2,Z)$-orbit) and the cross ratio lambda of the four branch values has been well-studied. More recently, in Minimal Surface Theory, it was also more convenient to assume that the branch values of a degree two elliptic functions were given and that the periods had to be computed. Moreover, symmetries became more important than in the earlier studies.

Note that the four branch points of a degree two elliptic function (also called "two-division points", or Zweiteilungspunkte) form a half-period lattice. There are three involutions of the torus which permute these branch points; each of these involutions has again four fixed-points and these are all midpoints between the four branch points. Since each of the involutions permutes the branch points, it transforms the elliptic function by a Moebius transformation. In Minimal Surface Theory, period conditions could be solved without computations if those Moebius transformations were not arbitrary, but rather were isometric rotations of the Riemann sphere—see in the Surface Category the minimal surfaces by Riemann and those named Jd and Je. This suggested the following construction: As degree two MAPS from a torus ($T = \mathbb{C}/\mathbb{L}$) to a sphere, we have the natural quotient maps $T/{-id}$; these maps have four branch points, since the 180 degree rotations have four fixed points. To get well defined FUNCTIONS we have to choose three points and send them to $\{0, 1, \infty\}$. We choose these points from the midpoints between the branch points, and the different choices lead to different functions. The symmetries also determine the points that are sent to $\{-1, +i, -i\}$. In this way we get the most symmetric elliptic functions, and they are denoted JD, JE, JF. The program allows one to compare them with Jacobi's elliptic functions. The function $WP = JE * JF$ has a double zero, a double pole and the values $\{+i, -i\}$ on certain midpoints (diagonal ones in the case of rectangular tori). Up to an additive and a multiplicative constant it agrees with the Weierstrass $\wp$-function, but in our normalization it is the Gauss map of Riemann's minimal surface on each rectangular torus.

We compute the J-functions as follows. If one branch value is called +B, then the others are $\{-B, +1/B, -1/B\}$. Therefore the function satisfies the differential equations

$$(J')^2 = (J'(0))^2(J^4 + 1 - (B^2 + 1/B^2)J^2),$$
$$J'' = (J'(0))^2(2J^3 - (B^2 + 1/B^2)J).$$

Numerically we solve this with a fourth order scheme that has the analytic continuation of the square root $J' = \sqrt{J'^2}$ built into it.

H.K.