Space Curves

in 3D-XplorMath, a Visualization Program

19.) Userdefined by Curvature and Torsion

20.) Other Userdefined Parametrized Space Curves

The Helix*

The helix is the simplest nonplanar space curve. It can be translated along itsself by a group of isometries (called *screw motions*) and therefore has its geometric invariants – the curvature and the torsion – constant.
Our (circular) helix as a parametrized curve c is given (with defaults $aa = bb = 1.5,\ cc = 0.25$) as

$$c(t) = (aa\cos(t),\ bb\sin(t),\ cc(t - tmin) - 3).$$

In the default `Morph` we extend the helix like pulling a bed spring and therefore want to keep its length constant. To do this we compute $f := (aa^2 + cc^2)^{-1/2}$ and show the reparametrized curve $c(f \cdot t)$.

Before we do the morph we select from the Action Menu `Show As Tube`. These tubes are either made with the 'Frenet Frame' or with a 'Parallel Frame'. The tube behaves like an elastic rod if we choose in the Action Menu `Parallel Frame`. The default morph now shows (at the right end, the left is kept fixed) that the tube also twists around itsself while it is extended. When this occurs with electrical wires or water hoses that are pulled sideways from their coil, it is a well known and annoying phenomenon.

H.K.

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

Torus Knots*

Torus knots are quite popular space curves because they represent the simplest way to write down knotted curves in $\mathbb{R}^3$. Our knots are parametrized as

$$c(t) := \begin{pmatrix} aa + bb \cdot \cos(dd \cdot t) \cdot \cos(ee \cdot t) \\ aa + bb \cdot \cos(dd \cdot t) \cdot \sin(ee \cdot t) \\ cc \cdot \sin(dd \cdot t) \end{pmatrix}$$

with defaults $aa = 3,\ bb = 1.5,\ cc = 1.5,\ \ dd = 5,\ ee = 2.$

The default `Morph` changes the torus size. If, before the morph, one chooses in the Action Menu `Show As Tube` and `Parallel Frame` then one notices that the twisting of the tube (see the ATO of the helix for more details) is clearly visible already for rather small changes of the shape of the torus.

The Action Menu has also the entry `Show Dotted Torus`. Selecting it adds the torus to the picture. This is more spectacular when viewing in `Anaglyph Stereo Vision`, through red/green filter glasses. Observe that our brain gets these several thousand dots sorted out into corresponding pairs of red and green dots that then form the torus surface in $\mathbb{R}^3$ - and this seems to happen instantly.

The best method to get a feeling for the curvature of a space curve is to select in the Action Menu `Show Osculating Circles & Evolute`. The Radius r of the circle

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is the *radius of curvature* of the curve at the current point and $\kappa := 1/r$ is called the *curvature* (at that point). The direction from $c(t)$ to the midpoint of the osculating circle determines always the direction of the second basis vector of the Frenet frame.

If one uses the Parallel Frame, then one has to represent the curvature by a vector of length κ in the plane spanned by the two normal vectors of the Parallel frame. If one has selected, in the Action Menu, `Parallel Frame` and clicks `Show Repére Mobile` then this curvature vector is drawn, together with its past history, in each normal plane. – The last entry in the Action Menu, `Show Frenet Integration` does the opposite: if the curvature vector function is given in the initial normal plane then the demo reconstructs the curve by integrating the Frenet equation.

H.K.

Genus Two Knots*

Torus knots, see the previous entry, are the most easily described knots and, in particular when viewed on a torus, they are also very easy to visualize.

If one wants to visualize other knots on some surface, one needs more complicated surfaces than tori. From this point of view the next simplest knots can be put on a genus 2 surface. The surface we chose looks like two tori which are joined by a small handle. (The size of these tori is controlled by the parameters aa and bb as for torus knots.) The surface is implicitly described by an equation (see implicit surfaces in the surface category) and can be made fatter by increasing ff. As examples of genus 2 knots we chose the connected sums of two (dd, ee) - torus knots. The sign of hh controls whether the two torus knots are connected with reflectional symmetry or with 180° rotational symmetry. The two simplest examples are the *Square Knot* and the *Granny Knot* where two (3, 2) - torus knots (=Trefoil Knots) are connected with the two types of symmetry.

The sum of the two torus knots is first constructed outside the surface, then projected onto the surface and finally smoothed with a length minimizing algorithm. The result is good enough for tubes made with a Parallel frame, whereas the tube from the Frenet frame is not smooth.

H.K.

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Cinquefoil Knot*

Parametric Formulas for the Cinquefoil Knot:

$$P.x := (2 - \cos(2\ t/(2\ aa + 1))) \cdot \cos(t);$$
$$P.y := (2 - \cos(2\ t/(2\ aa + 1))) \cdot \sin(t);$$
$$P.z := \quad - \sin(2\ t/(2\ aa + 1));$$

The choice $aa = 1$ gives a Trefoil knot, $aa = 2$ the Cinquefoil, and in general $aa = k$ gives the (2k+1)-foil knot (the program rounds aa before using it). The parameter range for t should be 0 to $(4\ k + 2)\ \pi$. If you change aa in the Set Parameters... dialog, then these values of tMin and tMax are set also, but you can change them later in the Set t,u,v Ranges... dialog.

A nice animation of the Cinquefoil knot can be obtained by first choosing `Show As Tube` from the Action menu, `Anaglyph Stereo Vision` from the View menu, and then `Rotate` from the Animation menu.

R.S.P.

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Trefoil Knot*

Parametric formulas for the Trefoil Knot:

$$x = 0.01\ (41\ \cos(t) - 18\ \sin(t) - 83\ \cos(2\ t) -$$
$$83\ \sin(2\ t) - 11\ \cos(3\ t) + 27\ \sin(3\ t)) \cdot hh$$
$$y = 0.01\ (36\ \cos(t) + 27\ \sin(t) - 113\ \cos(2\ t) +$$
$$30\ \sin(2\ t) + 11\ \cos(3\ t) - 27\ \sin(3\ t)) \cdot hh$$
$$z = 0.01\ (45\ \sin(t) - 30\ \cos(2\ t) + 113\ \sin(2\ t) -$$
$$11\ \cos(3\ t) + 27\ \sin(3\ t)) \cdot hh$$

The Trefoil knot, Figure 8 Knot, Granny Knot, Square Knot, displayed by 3D-XplorMath are all harmonic or Fourier knots. That is they are parametrized using finite Fourier series for all three coordinates. The particular parametrizations are taken from the 1995 PhD thesis of Aaron Trautwein at The University of Iowa.

Compare the rotation of the Frenet frame along this trefoil knot (defined with harmonic polynomials) and along the trefoil that results when you select `Torus Knot` with the parameters (dd=3, ee=2): Near the points on the torus knot where the curvature is very small, the rotation speed of the Frenet frame is large. - The Trefoil Knot can be shown with a Satellite Knot, default $dd = 55, ee = 2$.

In stereo mode a Möbius band bounded by the Trefoil Knot is added. Their handedness depends on the sign of hh. R.S.P.

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Figure 8 Knot, Granny Knot, Square Knot*

The Trefoil knot, Figure 8 Knot, Granny Knot, Square Knot, displayed by 3D-XplorMath are all harmonic or Fourier knots. That is they are parametrized using finite Fourier series for all three coordinates. The particular parametrizations are taken from the 1995 PhD thesis of Aaron Trautwein at The University of Iowa.
Satellite Knots can be added in the Action Menu to these four knots.

The Figure 8 Knot is an alternating prime knot with minimal crossing number 4. It is the next simplest knot after the Trefoil Knot.
Parametric formulas for the Figure 8 Knot:

$$\begin{aligned} x =& (32\cos(t) - 51\sin(t) - 104\cos(2t) - 34\sin(2t) + \\ & 104\cos(3t) - 91\sin(3t))/100 \\ y =& (94\cos(t) + 41\sin(t) + 113\cos(2t) - 68\cos(3t) - \\ & 124\sin(3t))/140 \\ z =& (16\sin(t) + 138\cos(2t) - 39\sin(2t) - 99\cos(3t) - \\ & 21\sin(3t))/70 \end{aligned}$$

The Granny Knot and the Square Knot are not prime, both are sums of two Trefoil Knots. The Square Knot has a mirror symmetry so that one Trefoil is left handed the

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other right handed. The Granny Knot is the sum of two same-handed Trefoil Knots.

Parametric formulas for the Granny Knot:

$$
\begin{aligned}
x =&(-22\cos(t) - 128\sin(t) - 44\cos(3t) - 78\sin(3t))/80\\
y =&(-10\cos(2t) - 27\sin(2t) + 38\cos(4t) + 46\sin(4t))/80\\
z =&(70\cos(3t) - 40\sin(3t))/100
\end{aligned}
$$

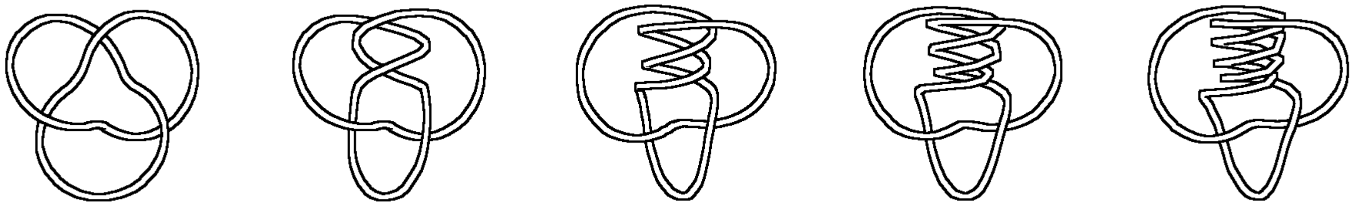
Parametric formulas for the Square Knot:

$$
\begin{aligned}
x =&(-22\cos(t) - 128\sin(t) - 44\cos(3t) - 78\sin(3t))/100\\
y =&(11\cos(t) - 43\sin(3t) + 34\cos(5t) - 39\sin(5t))/100\\
z =&(70\cos(3t) - 40\sin(3t) + 18\cos(5t) - 9\sin(5t))/100
\end{aligned}
$$

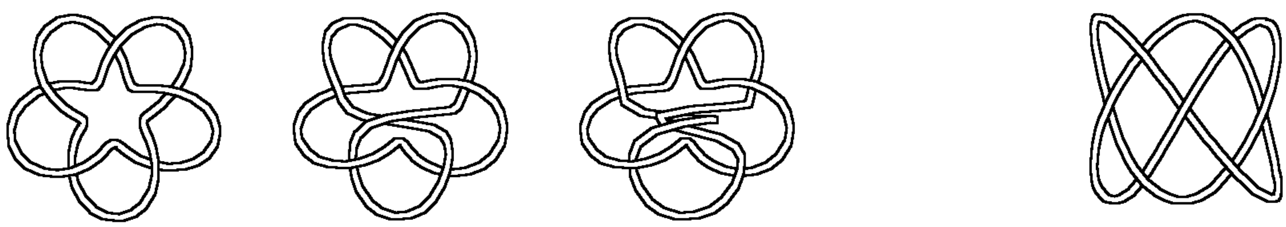
R.S.P.

Morph Through Five Prime Knots*

A prime knot is a knot that cannot be written as the knot sum of smaller knots. For example, the Square Knot and the Granny Knot are not prime since each is a sum of two Trefoil Knots. There are 14 prime knots with at most 7 minimal number of crossings. They have been hand drawn so often that they have assumed an esthetically defined standard shape. Of these first 14 prime knots the following ones are in a morphing family, the prime knots $3_1, 4_1, 5_2, 6_1, 7_2$. Choose $dd = 3$ and $0 \leq ff \leq 4.3$ in Set Morphing and the program will deform the Trefoil Knot through the following images:



If one chooses $dd = 5$ and $0 \leq ff \leq 2.3$ in Set Morphing then the program will deform the (5,2)-Torus Knot through the following images of the prime knots $5_1, 6_2, 7_5$:



The prime knot 7_4 is the default Lissajous space curve.

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There are 249 prime knots with at most 10 minimal number of crossings. One can visualize those via the Space Curves Menu entry: `V. Jones Braid List`.
The notion of prime knot is important because Horst Schubert proved that the decomposition of a knot as knot sum (= connected sum) of prime knots is unique. The knot invariants are a good way to check whether a given knot is a prime knot.
There is an easy sufficient criterion that guarantees that the knot under consideration cannot be drawn with fewer crossings. First we define *alternating* and *reduced alternating* knots: if the thread of the knot passes alternatingly through overcrossings and undercrossings then the knot is called **alternating**. For example, if we twist a circle into a figure 8 we obtain an alternating trivial knot. In this case we observe an easily recognizable property of the crossing in the knot diagram: if the crossing is removed the knot diagram decomposes into two components. A crossing with this property is called an *isthmus*. Clearly, one can always rotate one component of the knot diagram through 180 degrees, i.e. untwist and thereby remove the isthmus to obtain a representation with fewer crossings. An alternating knot without an isthmus is called a *reduced alternating knot*.

Theorem : Reduced alternating knots cannot be represented with fewer crossings, they are always non-trivial.

All prime knots with at most 7 crossings are reduced alternating knots. H.K.

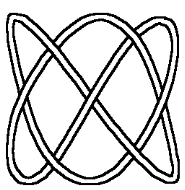
Lissajous Curves, e.g. the Prime Knot 7_4[*]

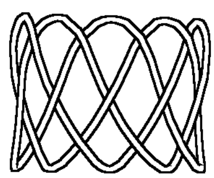
Lissajous curves are a popular family of planar curves, resp. space curves. They are complicated enough to be interesting, but regular enough to be esthetically pleasing. They are described by simple formulas:

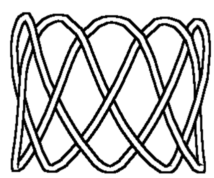
$$x(t) := aa \cdot \sin(2\pi \cdot dd \cdot t)$$
$$y(t) := bb \cdot \sin(2\pi \cdot ee \cdot t + gg)$$
$$z(t) := aa \cdot \sin(2\pi \cdot ff \cdot t + cc)$$

In 3DXM the parameters dd, ee, ff are rounded to integers so that the curves are closed on the interval $[0, 1]$. The default morph varies the phase gg from 0 to $\pi/2$. – The Lissajous curves are also physically interesting, they describe the joint motion of orthogonal uncoupled oscillators $(x(t), y(t), z(t))$ with different frequencies.

A **prime knot** is not a knot sum of smaller knots. E.g. Square Knot and Granny Knot are not prime: each is a sum of two Trefoil Knots. There are 14 prime knots with the minimal number of crossings ≤ 7, see the documentation `About This Object` for `V.Jones Braid List`. The 4th 7-crossings-knot, the prime knot 7_4, is our default Lissajous space curve, $(dd, ee, ff, gg) = (2, 3, 7, \pi/2)$. – Other alternating examples are: $(dd, ee, ff) = (2, 5, 13)$, $(4, 3, 23)$:







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There are 249 prime knots with at most 10 minimal number of crossings. One can visualize those via the Space Curves Menu entry: `V.Jones Braid List`.
The notion of prime knot is important because Horst Schubert proved that the decomposition of a knot as knot sum (= connected sum) of prime knots is unique. The knot invariants are a good way to check whether a given knot is a prime knot. There is no more elementary criterion to recognize a knot as prime.

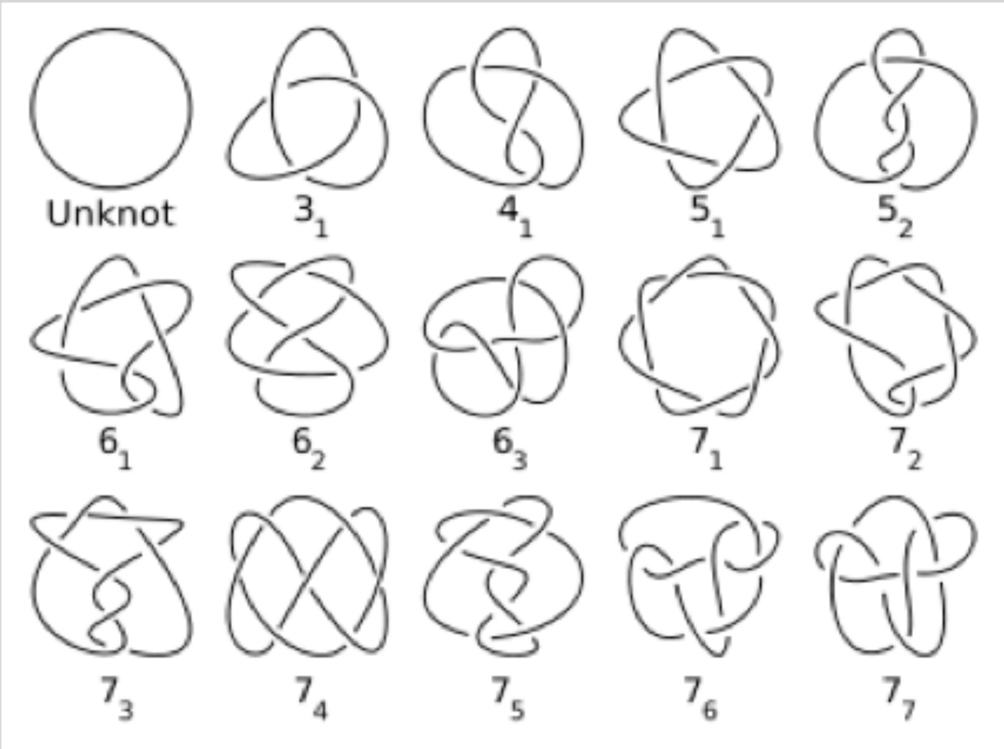
There is an easy sufficient criterion that guarantees that the knot under consideration cannot be drawn with fewer crossings. First we define *alternating* and *reduced alternating* knots: if the thread of the knot passes alternatingly through overcrossings and undercrossings then the knots is called **alternating**. For example, if we twist a circle into a figure 8 we obtain an alternating trivial knot. In this case we observe an easily recognizable property of the crossing in the knot diagram: if the crossing is removed the knot diagram decomposes into two components. A crossing with this property is called an *isthmus*. Clearly, one can always rotate one component of the knot diagram through 180 degrees, untwist and thereby remove the isthmus to obtain a representation with one less crossings. An alternating knot without an isthmus is called a *reduced alternating knot*.

Theorem : Reduced alternating knots cannot be represented with fewer crossings, they are always non-trivial.

H.K.

Braid List Of Prime Knots *

A prime knot is a knot that cannot be written as the knot sum of smaller knots. For example, the Square Knot and the Granny Knot are not prime since each is a sum of two Trefoil Knots. There are 249 prime knots with at most 10 minimal number of crossings. In 3DXM we use the braid representation of knots. Vaughn Jones gave this list to one of us in the 80s. The usual hand drawn versions are prettier than the braids:



A chart of all prime knots with seven or fewer crossings, not including mirror-images. (The unknot is not considered prime.)

Copied from the article 'Prime Knot' in Wikipedia.

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The above first 14 prime knots are all so called *alternating knots*: if one follows the thread of the knot then one passes – alternatingly(!) – overcrossings and undercrossings. A knot that is represented as a "reduced" alternating knot cannot be drawn with fewer crossings, in particular: a reduced alternating knot is always non-trivial. If one twists a circle to a figure 8 then one obtains a non-reduced alternating knot that is clearly trivial. Similarly, one can take the alternating `Granny Knot` in 3DXM and turn one of the Trefoil parts 180 degrees around the horizontal axis. One obtains an alternating knot with an additional crossing in the middle. Again, this knot is not reduced because by cutting out the new crossing the knot diagram decomposes into two components. Such an easily recognizable crossing is called an *isthmus*. One can always untwist an isthmus and obtain a knot with one less crossing. A knot diagram without an isthmus is called *reduced*.

The notion of prime knot is important because Horst Schubert proved that the decomposition of a knot as knot sum (= connected sum) of prime knots is unique.
All torus knots are prime knots. The genus 2 knots in 3DXM are sums of two torus knots.

The space curve "Morph Prime Knots 5 4 3" has a default morph that runs through the prime knots 3.1, 4.1, 5.2, 6.1, 7.2. If one changes dd from 3 to 5 then the ff-morph runs through 5.1, 6.2, 7.5. The prime knot 7.4 is shown as a Lissajous knot.

H.K.

The Intersection of Two Cylinders *

The image shows the space curve defined implicitly as the intersection of the two cylinders:

$$y^2 + z^2 = ff$$

and

$$(\cos(aa)x + \sin(aa)y)^2 + (z - cc)^2 = gg.$$

These two cylinders are made visible by displaying a random set of dots on each of them. In the Action Menu one can choose to put more random dots on the boundary of the intersection of the two solid cylinders.

In the default settings the two cylinders touch and the default morph rotates one of them by changing aa.

We find it interesting to change the radius of the smaller cylinder while the cylinders keep touching: morph gg up to ff while keeping $dd = 0$, since we compute (behind the user)

$$cc = \sqrt{ff} - \sqrt{gg} + dd.$$

At $gg = ff$ the intersection curve degenerates into two ellipses (for each aa).

The distance between the tangent planes of the two cylinders (at their common normal) is dd.

H.K.

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Userdefined Implicit Space Curves*

The exhibit shows the intersection curve of two surfaces, given by equations $F1(x, y, z) = ff,\ F2(x, y, z) = gg$.
To see also the surfaces (as dotted point clouds) choose the corresponding entry in the Action Menu.
The initial dialogue offers three different defaults for the surfaces given by $F1,\ F2$:
1.) A conic and a plane with the default morph tilting the plane.
2.) The graph of a function $\mathbb{R}^3 \mapsto \mathbb{R}$ and a cylinder. This exhibit can be used to explain extrema under side conditions.
3.) A torus and a tangent plane. This is an example where the intersection has double point singularities at those points where the intersection of the surfaces is not transversal.

By varying these defaults one can create a rich collection of space curves. (The number of points in the point clouds cannot be changed in this exhibit.)

H.K.

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About Spherical Curves*

In many ways there is a close analogy between planar Euclidean geometry and two-dimensional spherical geometry. In the ATO for spherical ellipses we translate the sum-of-distances definition from the plane to the sphere and use the same arguments as in the plane to construct points and tangents of the curve. Similarly, in the ATO about spherical cycloids, we roll spherical circles on spherical circles. Such analogies of course require basic notions which correspond to each other.

Lines and Triangles

Straight lines in the plane are the shortest connections between their points. On the sphere the shortest connections are great circle arcs that are not longer than half way around. A line cuts the plane into two congruent half-planes that are interchanged by the reflection in the line. Similarly, the sphere is cut by the plane of a great circle into two congruent half-spheres, and the reflection in the plane interchanges these two half-spheres. Therefore we speak of the reflection (of the sphere) in a great circle. These analogies are enough to translate the planar notion *straight line* to the spherical notion *great circle*. Three points A, B, C and three shortest connections of lengths a, b, c make a triangle — in the plane or on the sphere.

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The angles at the points (or vertices) are denoted α, β, γ. For the plane, the basic triangle formulas are close to the definition of sine and cosine:

Projection theorem:	$c = a \cdot \cos\beta + b \cdot \cos\alpha$
Sine theorem:	$b \cdot \sin\alpha = h_c = a \cdot \sin\beta$
Cosine theorem:	$c^2 = a^2 + b^2 - 2ab\cos\gamma$

Note that the more complicated third formula follows from the first two: Use the Sine theorem in the form $0 = b \cdot \sin\alpha - a \cdot \sin\beta$ and add the square of this to the square of the Projection theorem. Simplify with $\cos^2 + \sin^2 = 1$ and use the trigonometric identity $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta) = \cos(\pi-\gamma) = -\cos(\gamma)$ to obtain the Cosine theorem.

To derive similar formulas for spherical triangles, use geographic coordinates on the standard unit sphere, with the polar center at the north pole $C := (0,0,1)$. A point A at spherical distance b from C satisfies $\langle A, C\rangle = \cos b$. Thus, after rotation into the x-z-plane, it has coordinates $A := (\sin b, 0, \cos b)$. A third point B at distance a from the pole C and such that the angle $\angle ACB$ equals γ has spherical polar coordinates $B := (\sin a\cos\gamma, \sin a\sin\gamma, \cos a)$. The spherical cosine formula follows by taking a scalar product:

$$\langle A, B\rangle = \cos c = \cos a\cos b + \sin a\sin b\cos\gamma.$$

The name is justified since a Taylor approximation up to second order gives the corresponding formula for the plane.

For more details: note that the graph of the function $x \to \cos x$ lies above the graph of the quadratic function $x \to 1 - x^2/2$ and not above any wider parabola $x \to 1 - x^2/(2+\epsilon)$. Therefore $1 - x^2/2$ is called the quadratic Taylor approximation of cos near $x = 0$. We substitute this approximation for $x = a, x = b, x = c$, and similarly $\sin x \approx x$, in the spherical cosine formula and obtain:
$\cos c \approx 1 - c^2/2 \approx (1 - a^2/2)(1 - b^2/2) + ab\cos\gamma$, *or*
$c^2 \approx a^2 + b^2 - 2ab\cos\gamma$, *which is the planar formula.*

For a more systematic derivation we use the reflection R which interchanges C, A and observe that $R(B)$ has the coordinates $R(B) := (\sin c \cos\alpha, \sin c \sin\alpha, \cos c)$. But $R(B)$ can also be computed from the reflection matrix and the coordinates of B. Equating the two expressions gives three formulas between a, b, γ on one side and c, α on the other side. Of course these formulas hold for any permutation of A, B, C:

$$\begin{pmatrix} -\cos b & 0 & \sin b \\ 0 & 1 & 0 \\ \sin b & 0 & \cos b \end{pmatrix} \cdot \begin{pmatrix} \sin a \cos\gamma \\ \sin a \sin\gamma \\ \cos a \end{pmatrix} =$$

$$\begin{pmatrix} -\sin a \cos b \cos\gamma + \cos a \sin b \\ \sin a \sin\gamma \\ \cos a \cos b + \sin a \sin b \cos\gamma \end{pmatrix} = \begin{pmatrix} \sin c \cos\alpha \\ \sin c \sin\alpha \\ \cos c \end{pmatrix}.$$

We use for these formulas the same names as in the planar case since an even simpler Taylor approximation simplifies also the first two equations to their planar counterparts:

Projection theorem: $\sin c \cos\alpha =$
$$-\sin a \cos b \cos\gamma + \cos a \sin b$$
Sine theorem: $\sin c \cdot \sin\alpha = \sin a \cdot \sin\gamma$

Cosine thm: $\cos c = \cos a \cos b + \sin a \sin b \cos\gamma.$

A consequence of the first two theorems is the

Angle cosine: $\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cos c.$

Application: Platonic Solids

Two-dimensional spherical geometry captures certain aspects of three-dimensional Euclidean geometry. For example, if we project an *icosahedron* from its center to its circumsphere then the 20 triangular faces of the icosahedron are mapped to a tessellation of $\mathbb{S}^2$ by 20 equilateral triangles *whose angles are* 72° *because five triangles meet at every vertex*. From the angle cosine theorem we read off the edge-length σ of these triangles, with $\alpha = 2\pi/5$ we have for the

Icosahedron: $(\cos\alpha + \cos^2\alpha)/\sin^2\alpha = \cos\sigma.$

Given the above spherical tools this is a conceptually very simple construction.

Osculating Circles

At every point of a twice differentiable curve c on $\mathbb{S}^2$ one can determine its osculating circle: the parametrized circle that agrees with c up to the second derivative at that point. While it is easy to place a ruler next to a curve so that the ruler approximates a tangent line, one cannot so easily guess these best approximating circles. For all planar curves and space curves in 3DXM one can choose Osculating Circles from the Action Menu and one can believe that the resulting images show best approximating circles. In the case of spherical curves one observes that these osculating circles actually lie on the sphere. To understand this, consider the usual osculating circle in $\mathbb{R}^3$ and intersect its plane, the *osculating plane* of the curve c, with $\mathbb{S}^2$; this intersection circle is clearly a better approximation of the curve than any other circle in this plane and therefore it is the osculating circle. Although we cannot yet describe the *curvature* of a curve by a real valued function, we can already agree that, at each point, a space curve is curved as strongly as its osculating circle. We call the spherical radii of these circles the spherical curvature radii and we are ready to translate geometric constructions (with curves) from the plane to the sphere.

Parallel curves of a spherical curve c on $\mathbb{S}^2$. We define $\eta(t) := \dot{c}(t) \times c(t)/|\dot{c}(t)|$ as the oriented spherical unit normal of c. The parallel curve at spherical distance ϵ is then in complete analogy with the plane given as

$$\text{Parallel Curves on } \mathbb{S}^2: \; c_\epsilon(t) := \cos\epsilon \cdot c(t) + \sin\epsilon \cdot \eta(t).$$

It is easy to check that the curvature radii of c_ϵ are obtained by adding ϵ to the curvature radii of c — which is what our intuition expects of parallel curves.

Spherical Evolvents (also called involutes). For a physical realization of an evolvent attach a string segment to the curve and move the end point so that the string is always tangent to the curve, in the forward or in the backward direction. The Euclidean formula for the backwards evolvent is (assuming $|\dot{c}(t)| = 1$)

$$e(t) := c(t) - (t - t_0) \cdot \dot{c}(t),\ t \geq t_0.$$

A remarkable property of the evolvent is that $t - t_0$ is its curvature radius at $e(t)$.

We translate this construction to the sphere. The formula for the spherical evolvent is (assume again $|\dot{c}(t)| = 1$)

$$e(t) := \cos(t - t_0) \cdot c(t) - \sin(t - t_0) \cdot \dot{c}(t).$$

A short computation shows that the spherical curvature radius at $e(t)$ is $t - t_0$, as in the plane. Also, it is true for the plane and for the sphere that the segment from $c(t)$ to $e(t)$ is orthogonal to $\dot{e}(t)$, i.e., this segment is the curvature radius of the evolvent at $e(t)$.

Spherical Evolutes. For any given (planar or) spherical curve c we call the curve of the (planar or) spherical midpoints of the osculating circles of c the *(planar or) spherical evolute* of c. In 3DXM this can best be seen in the demo for *Spherical Ellipses*. In the previous paragraph we have seen that, in the plane and on the sphere, the evolvent of the evolute of c is this given curve c. Thus, the natural translations of notions from the plane to the sphere continue to have natural properties.

What is Curvature?

More precisely, what real number should measure the size of the curvature at one point of the curve c, and which real valued function should describe the curvatures of c? For the plane, differential geometers have agreed to take the rotation speed of a unit normal of c as the quantitative size of its curvature. For example, the rotation speed of the unit normal n of a circle of radius r (use arc length parametrization) is $1/r$, since $c(t) = r \cdot (\cos(t/r), \sin(t/r))$, $|\dot{c}(t)| = 1$ and $n(t) = (\cos(t/r), \sin(t/r))$, hence $\dot{n}(t) = (1/r) \cdot \dot{c}(t)$. Although this is a good reason for taking $1/r$ as the curvature of a circle of radius r in the plane, the argument does not carry over to $\mathbb{S}^2$, since: *What is the spherical rotation speed of the spherical normal?* Of course we could also call on the sphere $1/$*curvature radius* the curvature of the curve. This is not a good idea on $\mathbb{S}^2$ since circles of radius $\pi/2$ are great circles, i.e., shortest connections, and we would expect them to have curvature 0. Fortunately, there is for the plane another good reason for taking $1/r$ as "the" curvature, and this time the corresponding computation can be repeated on $\mathbb{S}^2$. If we imagine a family of parallel curves then it looks as if the length grows faster if the curvature is larger.

We can make this intuition more precise with a computation. First, in the plane:

$$
\begin{aligned}
&c_\epsilon(t) := c(t) + \epsilon \cdot n(t), \ \{\dot{c}(t), n(t)\} \text{ orthonormal} \\
&\dot{c}_\epsilon(t) = \dot{c}(t) + \epsilon \cdot \dot{n}(t), \ \ \dot{n}(t) = \kappa(t) \cdot \dot{c}(t) \\
&\frac{d}{d\epsilon} |\dot{c}_\epsilon(t)|_{\epsilon=0} / |\dot{c}(t)| = \kappa(t).
\end{aligned}
$$

Here, the second line defines the curvature as the rotation speed of the normal and the third line says *that this curvature function can also be computed as the change of length of tangent vectors in a parallel family of curves.* Of course we can do the same computation as in line three for spherical curves:

$$
\begin{aligned}
&c_\epsilon(t) := \cos\epsilon \cdot c(t) + \sin\epsilon \cdot \eta(t)\\
&\dot c_\epsilon(t) = \cos\epsilon \cdot \dot c(t) + \sin\epsilon \cdot \dot\eta(t)\\
&\dot\eta(t) = \ddot c(t) \times c(t)\\
&|\dot c_\epsilon(t)|/|\dot c(t)| = \langle \dot c_\epsilon(t), \dot c(t)\rangle / \langle \dot c(t), \dot c(t)\rangle =\\
&\ \cos\epsilon + \sin\epsilon \langle \dot\eta(t), \dot c(t)\rangle / \langle \dot c(t), \dot c(t)\rangle\\
&\frac{d}{d\epsilon}|\dot c_\epsilon(t)|_{\epsilon=0}/|\dot c(t)| = -\langle \eta(t), \ddot c(t)\rangle / \langle \dot c(t), \dot c(t)\rangle.
\end{aligned}
$$

Before we take this as the definition of spherical curvature for spherical curves we check which function of the radius we get for circles of spherical radius r:

$$
\begin{aligned}
&c_r(t) = (\sin r \cos t, \sin r \sin t, \cos r)\\
&\eta(t) = \frac{d}{dr} c_r(t) = (\cos r \cos t, \cos r \sin t, -\sin r)\\
&\ddot c_r(t) = -(\sin r \cos t, \sin r \sin t, 0), \qquad \text{finally:}\\
&-\langle \eta(t), \ddot c(t)\rangle / \langle \dot c(t), \dot c(t)\rangle = \frac{\sin r \cos r}{\sin^2 r} = \cot r.
\end{aligned}
$$

This is a satisfying answer, since $\cot r$ behaves like $1/r$ for small r and $\cot(r = \pi/2) = 0$ as we expect for great circles.

Now we are ready for the definition and we remark that the historical name is *geodesic curvature*, not the more naive spherical curvature which we used above.

Definition. The geodesic curvature $\kappa_g(t)$ of a spherical curve $c(t)$ with spherical unit normal $\eta(t)$ is

$$\kappa_g(t) := -\langle \eta(t), \ddot{c}(t)\rangle / \langle \dot{c}(t), \dot{c}(t)\rangle.$$

The Spherical Frenet Equation

Finally we observe that for a unit speed spherical curve c we have the following natural orthonormal frame along the curve:

$$(e_1(t), e_2(t), e_3(t)) := (\dot{c}(t), c(t), \eta(t)),$$

and the geodesic curvature controls the derivative of this frame via the following *spherical Frenet equation*:

$$\begin{aligned}
\frac{d}{dt}\dot{c}(t) &= \qquad\qquad\qquad\quad -1 \cdot c(t) - \kappa_g(t) \cdot \eta(t) \\
\frac{d}{dt}c(t) &= \quad +1 \cdot \dot{c}(t) \\
\frac{d}{dt}\eta(t) &= +\kappa_g(t) \cdot \dot{c}(t)
\end{aligned}$$

Observe that the coefficient matrix

$$\begin{pmatrix} 0 & -1 & -\kappa_g \\ 1 & 0 & 0 \\ \kappa_g & 0 & 0 \end{pmatrix}$$

is skew symmetric. This fact implies that any solution $(e(t), f(t), g(t))$ with *orthonormal* initial conditions stays orthonormal. This says that $t \to f(t)$ is a spherical curve parametrized by arclength (namely: $|\dot{f}(t)| = |e(t)| = 1$). Moreover $g(t)$ is orthogonal to $f(t), \dot{f}(t)$ and therefore the spherical unit normal of f. The third Frenet equation says that the given function $\kappa_g(t)$ (because of $\dot{g}(t) = \kappa_g(t) \cdot e(t)$) is indeed the geodesic curvature of the curve $t \to f(t)$: to any given $\kappa_g(t)$ we have found a curve with that geodesic curvature.

We repeat: from elementary distance and triangle geometry to the differential geometry of curves we have explained a very close analogy between the Euclidean plane and the sphere. The 3DXM demos try to emphasize this.

HK.

Loxodrome*

A Loxodrome (also called a rhumb line) is a route that a boat would take if it kept a constant compass heading (so that on a Mercator projection it is simply a straight line). To be more formal, a loxodrome is a path that lies on the unit sphere in $\mathbb{R}^3$ and that makes a constant angle with the great circles of longitude (i.e. the meridians). Thus the loxodromes are analogous to the logarithmic spirals in the (complex) plane, which make a constant angle with the rays through the origin. In fact, since stereographic projection from the complex plane to the unit sphere is conformal (in other words: angle-preserving) and since the stereographic projection of the radial lines in the plane are the circles of longitude, it follows that the loxodromes are given by stereographically projecting the logarithmic spirals. On the other hand, since the exponential map of the complex plane to itsself is conformal and maps the lines parallel to the real axis to radial lines, it follows that the logarithmic spirals are just the images under the exponential map of straight lines, i.e. the images of $t \mapsto (aa + i) \cdot t$.

Hence we can define the loxodromes parametrically by

$$t \mapsto \mathrm{StereographicProjection}(\exp((aa + i) \cdot t)).$$

Note that the osculating circles all lie on the sphere.

R.S.P.

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

The Viviani Curve *

The Viviani curve is the intersection of a sphere of radius $2 \cdot aa$ and a cylinder of radius aa that touch at a single point, the double point of the curve. Parametric formulas for it are:

$$z = aa\ (1 + \cos(t)) = aa\ 2\ \cos(t/2)^2,$$
$$y = aa\ \sin(t) = aa\ 2\ \sin(t/2)\ \cos(t/2), \text{ and}$$
$$x = aa\ 2\ \sin(t/2)$$

Implicit equations for the two intersecting surfaces are:

$$x^2 + y^2 + z^2 = 4\ aa^2, \quad \text{a sphere of radius } 2\ aa,$$
$$(z - aa - bb)^2 + y^2 = aa^2, \quad \text{a cylinder of radius } aa.$$

The planar projections of this curve are therefore in general curves of degree 4, but because of its symmetries the Viviani curve has two orthogonal two-to-one projections that are simpler; namely curves of degree 2. Indeed projecting it to the y-z-plane we get a twice covered circle (use Settings Menu: Set Viewpoint and Up Direction 200,0,0), projecting to the x-z-plane gives a twice covered parabolic piece, $(1 - z/(2aa)) = (x/(2aa))^2$, while the projection to the x-y-plane is the degree 4 figure 8 with the equation (for $aa = 1/2$): $x^2 - y^2 = x^4$.
Note that the osculating circles lie on the sphere.

R.S.P.

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http://3D-XplorMath.org/

About Spherical Cycloids*

See also the ATOs for Spherical Ellipses and for Planar Rolling Curves, e.g. Astroid, Cardioid

Spherical Definition
(in analogy to planar case)

The spherical ellipses demonstrated already how definitions from planar Euclidean geometry can be repeated on the sphere; the demo illustrates that also spherical evolutes are analogous to the planar ones. Rolling curves, spherical cycloids, provide more such examples: simply let one spherical circle *roll* (on the inside or the outside) along another spherical circle. Here *roll* means that the arclengths (= angle at the center times sine of the spherical radius) of corresponding arcs of the two circles agree. The true rolling curves are obtained by looking at the curve traced out by one point of the rolling circle, but, just as in the plane, one may also look at the traces of other points on a fixed radius, inside or outside the rolling circle — choose bb different from 1 in the Settings Menu, Set Parameters Dialog.

The rolling construction is illustrated by choosing *Show Rolling Circle* in the Action Menu.

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Rolling curves have a very simple tangent construction. The point of the rolling circle which is in contact with the base curve has velocity zero – just watch cars going by. This means that the connecting segment (which is a piece of a great circle of the sphere) from this point of contact of the wheel to the endpoint of the (great circle) drawing stick is the (great circle) radius of the momentary rotation. The tangent of the curve drawn by the drawing stick is therefore orthogonal to this momentary radius. The 3DXM-demo draws the rolling curve and shows its tangents.
One can observe, for all spherical curves (in 3DXM: Viviani, Spherical Ellipses, Spherical Cycloids, Loxodrome), that the osculating circles lie on the sphere of the spherical curve by choosing *Show Osculating Circle* in the Action Menu. To understand this, note, that the osculating circle lies in the osculating plane (Action Menu!) and, of course, no circle in a given osculating plane can be a better approximation of the curve than the intersection of this plane with the sphere on which the curve lies.

H.K.

About Spherical Ellipses*

See ATO for Planar Ellipses

In 3DXplorMath the *Default Morph* shows a family of ellipses with fixed focal points F_1, F_2 as the larger axis aa varies from its allowed minimum $e = bb/2$ to its allowed maximum $\pi - e = \pi - bb/2$. Another interesting morph is $0.11 \leq aa \leq 1.43, \quad 0.2 \leq bb \leq \pi - 0.2$: the distance of the focal points increases until they are almost antipodal and the major axis is only slightly longer than the distance of the focal points.

Elementary Definition. Many elementary constructions from planar Euclidean geometry have natural analogues on the twodimensional sphere $\mathbb{S}^2$. For example, we can take the definition of planar ellipses and use it on the sphere as follows: Pick two points $F_1, F_2 \in \mathbb{S}^2$ of spherical distance $2e := dist(F_1, F_2) < \pi$ and define the set of points $P \in \mathbb{S}^2$ *for which the sum of the distances to the two points* F_1, F_2 *equals a constant* $=: 2a$, i.e. the set:

$$\{P \in \mathbb{S}^2;\ dist(P, F_1) + dist(P, F_2) = 2a\},$$

to be a Spherical Ellipse.

In the Euclidean plane there is only one restriction between the parameters of an ellipse: $2e < 2a$. Since distances on

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$\mathbb{S}^2$ cannot be larger than π we have two restrictions in spherical geometry: $2e < 2a < 2\pi - 2e$.

For fixed focal points, i.e. for fixed e, these curves cover the sphere (we allow that the smallest and the largest ellipse degenerate to great circle segments). One observes that the ellipse with $2a = \pi$ is a great circle and that ellipses with $2a > \pi$ are congruent to ellipses with $2a < \pi$ and focal points $-F_1, -F_2$.
This is because $dist(P, F) = \pi - dist(P, -F)$ implies

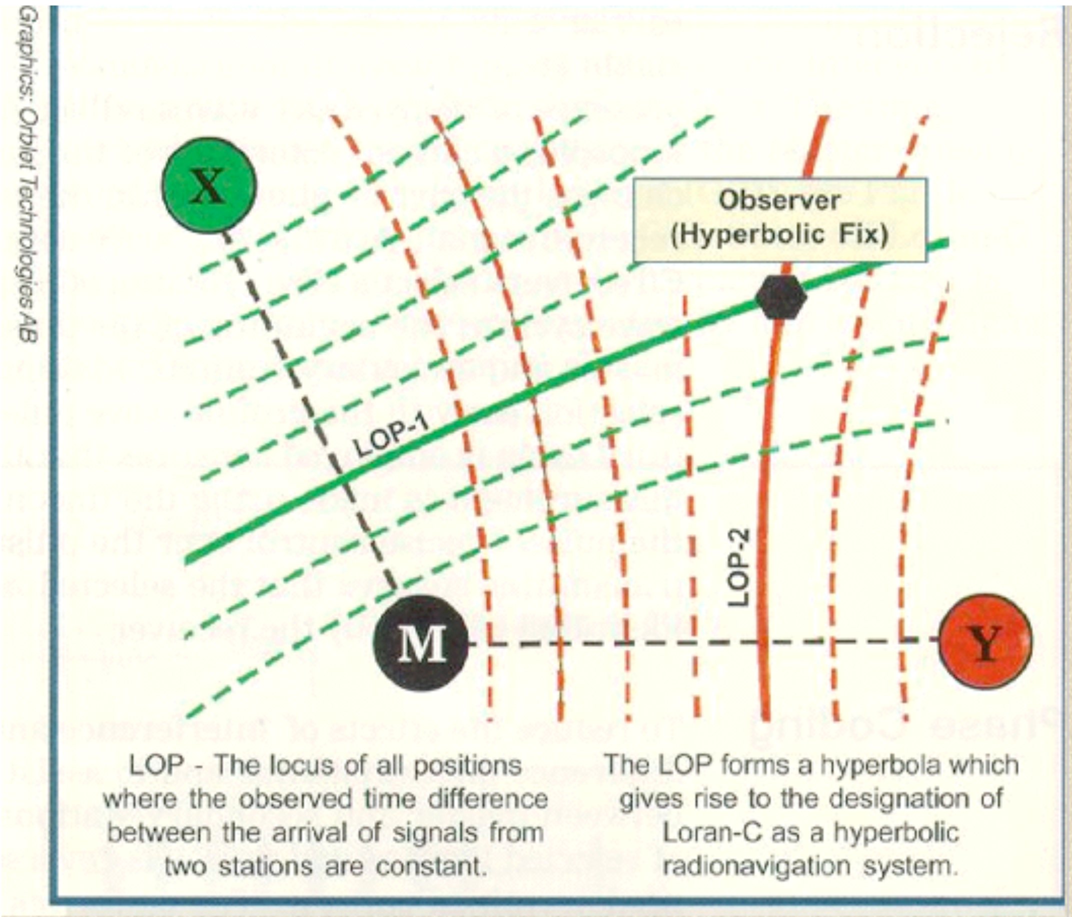
$$\pi < 2a = dist(P, F_1) + dist(P, F_2) \Rightarrow$$
$$dist(P, -F_1) + dist(P, -F_2) = 2\pi - 2a < \pi.$$

Similarly, on the sphere one does not need to distinguish between ellipses and hyperbolas:

$$\{P \in \mathbb{S}^2;\ dist(P, F_1) + dist(P, F_2) = 2a\} =$$
$$\{P \in \mathbb{S}^2;\ dist(P, F_1) - dist(P, -F_2) = 2a - \pi\}.$$

PRACTICAL APPLICATION. These curves are used since more than 50 years in the LORAN System to determine the position of a ship on the ocean as follows. Consider a pair of radio stations which broadcast synchronized signals. If one measures at any point P on the earth the time difference with which a pair of signals from the two stations arrives, then one knows the difference of the two distances from P to the radio stations. Therefore sea charts were prepared which show the curves of constant

difference of the distances to the two radio stations. This has to be done for several pairs of radio stations. In araes of the ocean where the families of curves (for at least two pairs of radio stations) intersect reasonably transversal it is sufficient to measure two time differences, then a look on the sea chart will show the ship's position as the intersection point of two curves, two sperical hyperbolas. On the site
http://webhome.idirect.com/...
~ jproc/hyperbolic/index.html or
~ jproc/hyperbolic/lorc_hyperbola.jpg
this is explained by the following map:



Elementary Construction, 3DXM-demo

Begin by drawing a circle of radius $2a$ around F_1 (called *Leitkreis* in German). Next, for every point C on this circle we find a point X on the spherical ellipse as follows: Let M be the midpoint of the great circle segment from C to F_2 and let T be the great circle through M and perpendicular to that segment. In other words, T is the symmetry line between C and F_2. Finally we intersect T with the *Leitkreis* radius from F_1 to C in X. — Because we used the symmetry line T we have $dist(X,C) = dist(X,F_2)$ and therefore:

$$\begin{aligned} dist(X,F_1) + dist(X,F_2) &= dist(X,F_1) + dist(X,C) \\ &= dist(C,F_1) = 2a. \end{aligned}$$

It is easy to prove that the great circle T is tangent to the ellipse at the point$X$.

Connection with Elliptic Functions

We met a family of ellipses all having the same focal points ('confocal') and also the orthogonal family of confocal hyperbolas in the visualization of the complex function $z \to z + 1/z$. In the same way two orthogonal families of confocal spherical ellipses show up in the visualization of elliptic functions from *rectangular tori* to the Riemann sphere (choose in the Action Menu: *Show Image on Riemann Sphere* and in the View Menu: *Anaglyph Stereo Vision*). — Note that in the plane all such families of confocal ellipses and hyperbolas are essentially the same, they differ only in scale. On the sphere we get different families for different rectangular tori, i.e. for different quadrupels of focal points $\{F_1, F_2, -F_1, -F_2\}$.

An Equation for the Spherical Ellipse

Abbreviate $\alpha := dist(X, F_1)$, $\beta := dist(X, F_2)$. The definition of a spherical ellipse says:

$$\cos(2a) = \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$
$$\text{with } \cos\alpha = \langle X, F_1\rangle,\ \cos\beta = \langle X, F_2\rangle.$$

We want to write the equation in terms of the scalar products which are linear in X. Therefore we replace $\sin^2 = 1 - \cos^2$ to get:

$$(1-\cos^2\alpha)(1-\cos^2\beta) = (\cos\alpha\cos\beta - \cos(2a))^2$$

or

$$1 - \cos^2\alpha - \cos^2\beta = -2\cos(2a)\cos\alpha\cos\beta + \cos^2(2a)$$

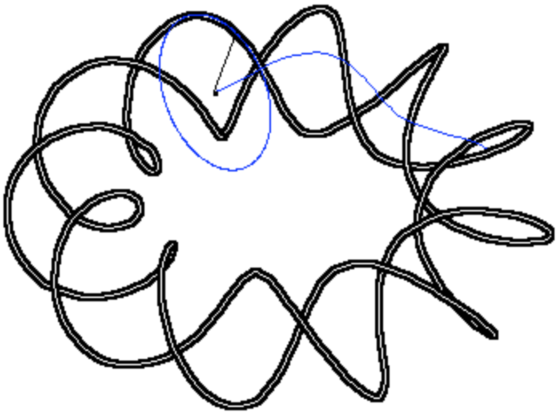
or, by replacing the cosines by the scalar products:

$$\sin^2(2a)\langle X, X\rangle - \langle X, F_1\rangle^2 - \langle X, F_2\rangle^2 =$$
$$-2\cos(2a)\cdot\langle X, F_1\rangle\cdot\langle X, F_2\rangle.$$

Observe that this is a homogenous quadratic equation in $X = (x, y, z)$. In other words: Our spherical ellipse is the intersection of the unit sphere with a quadratic cone whose vertex is at the midpoint of the sphere. So we get the surprisingly simple result: If one projects a spherical ellipse from the midpoint of the sphere onto some plane then one obtains a (planar) conic section.

H.K.

Space Curves of Constant Curvature *



2 - 11 Torus Knot of constant curvature.

See also:
About Spherical Curves

Definition via Differential Equations. Space Curves that 3DXM can exhibit are mostly given in terms of explicit formulas or explicit geometric constructions. The differential geometric treatment of curves starts from such examples and defines geometric properties, i.e., properties which do not change when the curve is transformed by an isometry (= distance preserving map, also called a rigid motion) of Euclidean space $\mathbb{R}^3$. The most important such properties are the curvature function κ and the torsion function τ. Once they have been defined one proves the *Fundamental Theorem of Space Curves*, which states that for any given continuous functions κ, τ there is a space curve with these curvature and torsion functions, and, that this curve is uniquely determined up to a rigid motion.

To define curvature, observe that at each point of a parametrized space curve $c(t)$ there is a parametrized circle $\gamma(t)$

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with

$$c(t_0) = \gamma(t_0), \dot{c}(t_0) = \dot{\gamma}(t_0), \ddot{c}(t_0) = \ddot{\gamma}(t_0).$$

This circle – which may degenerate to a straight line – is called the *osculating circle at* t_0, its radius is called *curvature radius at* t_0 and the inverse of the radius is called *the curvature at* t_0, $\kappa(t_0)$. The computation of curvature is simpler if the curve is parametrized by arc length, i.e. if the length of all tangent vectors is one, $|\dot{c}(t)| = 1$. One gets $\kappa(t) = |\ddot{c}(t)|$. Check this for the circle (arclength parametrization) $c(t) := r \cdot (\cos(t/r),\ \sin(t/r))$. The most common way to proceed is to assume that $\kappa(t) > 0$. This allows one to define the *Frenet basis* along the curve:

$$\begin{aligned} e_1(t) &:= \dot{c}(t), \\ e_2(t) &:= \ddot{c}(t)/\kappa(t), \\ e_3(t) &:= e_1(t) \times e_2(t). \end{aligned}$$

The Frenet basis defines three curves $t \mapsto e_j(t)$ on the unit sphere. To emphasize the fact that $e_j(t)$ are to be considered as vectors, not as points, one calls the length of their derivative, $|\dot{e}_j(t)|$, *angular velocity* or *rotation speed* and not just velocity. For example, the formula $\ddot{c}(t) = \kappa(t)e_2(t)$ says that $\kappa(t)$ is the rotation speed of $\dot{c}(t)$. Next, we get from $\dot{e}_1(t) \sim e_2(t)$ that the derivative of $e_3(t)$ is proportional to $e_2(t)$. This proportionality factor, the rotation speed of $e_3(t)$, is called the torsion function $\tau(t)$ of the curve $c(t)$. In formulas: $\tau(t) := \langle \dot{e}_3(t),\ e_2(t) \rangle$.

Now one changes the point of view and considers the two functions κ, τ as given. This turns the equations that were originally definitions of κ and τ into differential equations, the famous

$$\begin{aligned}
&\textit{Frenet-Serret Equations:}\\
\dot e_1(t) &= \phantom{-\kappa(t)\cdot e_1(t)-{}}\kappa(t)\cdot e_2(t),\\
\dot e_2(t) &= -\kappa(t)\cdot e_1(t) - \tau(t)\cdot e_3(t),\\
\dot e_3(t) &= \phantom{-\kappa(t)\cdot e_1(t)-{}}\tau(t)\cdot e_2(t),
\end{aligned}$$

or, with $\vec\omega(t) := -\tau(t)\cdot e_1(t) + \kappa(t)\cdot e_3(t)$,

$$\dot e_j(t) = \vec\omega(t)\times e_j(t).$$

Finally $\dot c(t) = e_1(t)$.

For given continuous functions κ, τ these differential equations have — for given orthonormal initial values — unique orthonormal solutions $\{e_1(t), e_2(t), e_3(t)\}$.
The curve $c(t) := \int^t e_1(s)ds$ is then parametrized by arc length and has the given curvature functions κ, τ.

The simplest curves in the plane, straight lines and circles, have constant curvature. One may wonder what constant curvature curves look like in $\mathbb{R}^3$. In 3DXM we illustrate the use of the Frenet-Serret equations by showing the following family of constant curvature curves:

$$\kappa(t) := aa,$$
$$\tau(t) := bb + cc\cdot\sin(t) + dd\cdot\sin(2t) + ee\cdot\sin(3t).$$

The function τ is, if $bb = 0$, skew symmetric at its zeros at 0 and π. This implies that the solution curves are symmetric with respect to the normal planes at these points. From this it follows that we can get *closed nonplanar curves of constant curvature* easily: the only requirement is that the angle of the normal planes at $c(0)$ and $c(\pi)$ has to be a rational multiple of π. *Every* $bb = 0$ *one-parameter family of examples in 3DXM therefore contains many closed examples* — select in the Animation Menu the default morph.

In the less symmetric case $bb \neq 0$ (but $dd = 0$) the function τ is even at the maxima and the minima, at $t = \pi/2,\ t = 3\pi/2$, and this implies that 180° rotation around e_2 at these points is also a symmetry of solution curves. This can be used to find more closed curves by solving 2-parameter problems as follows: For every value of aa, bb use cc to make the distance between the normals 0. Now change aa or bb slowly (continuing to use cc for keeping the distance between the normals 0) and observe how the angle between the symmetry normals varies. If this angle hits a value $2k/n \cdot \pi$ then n copies of the computed piece fit together to a smoothly closed curve.
If one has selected 'Constant Curvature' in the Menu 'Space Curves' then there is in the Action Menu an entry 'Other Closed Curves'. It opens a submenu where one can select first $bb = 0$ examples which are also hit by the default morph. Then there are $bb \neq 0$ embedded examples, some of them knotted. Moreover, the 11-2-knot has nonvanishing torsion and strongly resembles a torus knot. This is no coincidence since one can find constant curvature curves on tori by solving a second order ODE, and it is again a 2-parameter problem to close these up. – The example 'like 6 helices' looks in another way as one would imagine constant curvature curves: made up of left winding and

right winding pieces of helices.
Do not miss to select 'Show Osculating Circles & Evolute'. The constant radius of the osculating circles shows the constant curvature and the rotating motion of the radius shows size and sign of the torsion.

In 3DXM one can choose in the Action Menu 'Parallel Frame'.This frame is designed to rotate as little as possible along the curve, in $\mathbb{R}^3$. This property is more obvious when one looks at the torus knots than at the constant curvature curves. For further details see curves of *constant torsion*. The main advantage of these parallel frames is that they neither make it neccessary to assume more than continuity of the second derivative $\ddot{c}$, nor that $\kappa > 0$ everywhere, even straight lines are not exceptional curves if one works with these frames. Their differential equation is also simple:

Frenet-Serret Equations for Parallel Frames:

$$\begin{aligned}
\dot{e}_1(t) &:= \qquad\quad a(t)\cdot e_2(t) + b(t)\cdot e_3(t),\\
\dot{e}_2(t) &:= -a(t)\cdot e_1(t),\\
\dot{e}_3(t) &:= -b(t)\cdot e_1(t).
\end{aligned}$$

With an antiderivative $T(t)$ of the torsion $\tau(t) = T'(t)$ we can of course write the twodimensional curvature vector $(a(t), b(t))$ in terms of $\kappa(t), \tau(t)$, namely:

$$(a(t), b(t)) := \kappa(t)\big(\cos(T(t)), \sin(T(t))\big).$$

H.K.

About Space Curves of Constant Torsion *

See also: About Space Curves of Constant Curvature

Definition via Differential Equations

Most Space Curves that 3DXM can exhibit are given in terms of explicit formulas or explicit geometric constructions. In “About Space Curves of Constant Curvature” we explain how *curvature* and *torsion* of a space curve are defined. The definition immediately translates into a construction of the curve from curvature and torsion via the following *differential equations*, the famous

Frenet-Serret Equations:

$$\begin{aligned}
\dot e_1(t) &:= \qquad\quad \kappa(t)\cdot e_2(t),\\
\dot e_2(t) &:= -\kappa(t)\cdot e_1(t) - \tau(t)\cdot e_3(t),\\
\dot e_3(t) &:= \qquad\quad \tau(t)\cdot e_2(t).
\end{aligned}$$

For given continuous functions κ, τ these differential equations have — for given orthonormal initial values — unique orthonormal solutions $\{e_1(t), e_2(t), e_3(t)\}$. The curve $c(t) := \int^t e_1(s)ds$ is then parametrized by arc length and has the given curvature functions κ, τ.

The simplest curves in the plane are straight lines and circles, curves of constant curvature. It is therefore natural to discuss also space curves of constant curvature. In 3DXM

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we illustrate these by finding closed examples in the following family:

$$\kappa(t) := aa,$$
$$\tau(t) := bb + cc \cdot \sin(t) + dd \cdot \sin(2t) + ee \cdot \sin(3t).$$

To understand the Frenet-Serret equations better one can also study other special cases. Experimentation shows that the following curves of constant torsion

$$\kappa(t) := bb + cc \cdot \cos(ff \cdot t) + dd \cdot \cos(2ff \cdot t) + ee \cdot \cos(3ff \cdot t)$$
$$\tau(t) = aa$$

have an amusingly strong change of shape as one changes the parameters. Again we look for closed examples with the help of symmetries. Note that $180°$ rotations around the principal normals $e_2(t)$ at $t/ff = k\pi, k \in \mathbb{Z}$ are isometries of the curves. At $t/ff = \pi/2 + k\pi, k \in \mathbb{Z}$ the $180°$ rotations around the other normal vector of the frame, $e_3(t)$, are also isometries of the space curve. This allows us to formulate the *closing condition*:
If the normals $e_2(0)$ at $c(0)$, $e_3(\pi \cdot ff/2)$ at $c(\pi \cdot ff/2)$ intersect and if their angle is a rational multiple of π then the space curve closes up. Numerically one can use the parameter cc to keep the angle constant, e.g. at $\pi/3, \pi/4$, and then use aa to let the normals intersect. There are many closed solutions. Typically they look like a collection of bed springs which are joint by fairly straight pieces. If one allows these bed springs to have many turns then the closing values of aa and cc are almost equidistant. The default morph of 3DXM shows this, it contains two closed

and three approximately closed curves which are made of *three* bed springs with an increasing number of turns. It is easy to extend this family to springs with more turns, but one can also find all the small values, down to just one half turn for each spring. — We found no closed curves made of only *two springs*.

Here is a list of numerically closed curves:

Curves with 3-fold symmetry, $ff = 0.208$,

$$\begin{array}{cccc}
aa, & 0.178632213, & 0.284031845, & 0.417033334, \\
cc, & 0.2874008, & 0.90658882, & 2.19234962, \\
aa, & 0.513441035, & 0.59263462, & 0.628044, \\
cc, & 3.489480574, & 4.7901189, & 5.4411264, \\
aa, & 0.661324546, & 0.69281176, & 0.7227614 \\
cc, & 6.09244336, & 6.7440016, & 7.39575343
\end{array}$$

Curves with 4-fold symmetry, $ff = 0.23$,

$$\begin{array}{cccc}
aa, & 0.2137654757, & 0.3704887, & 0.479019355, \\
cc, & 0.234123448, & 0.89640923, & 1.59595534, \\
aa, & 0.56642393, & 0.6414483533, & 0.7081321561, \\
cc, & 2.30473675, & 3.01756515691, & 3.732639742, \\
aa, & 0.76871766, & 0.8246012, & 0.87671763 \\
cc, & 4.449136, & 5.1666082, & 5.8847911
\end{array}$$

Curve with 5-fold symmetry, $ff = 0.2324$,
$aa = 0.73855871446286$, $cc = 2.96466$

If ones does not begin with the differential equation but starts from the curve, then one cannot define the torsion at points where the curvature vanishes. This problem is caused by the use of the Frenet frame. Another frame is suggested by a mechanical consideration: If a massive sphere would move along the space curve (imagine the space curve as a wire and the sphere with a hole through which the wire slides without friction) then inertia would make the sphere avoid unnecessary rotations around the wire. In other words: A frame which is attached to the sphere so that it is normal to the wire remains normal and the derivatives of the normal vectors have *no normal components*. Such frames are called "parallel as normal vectors", or simply "parallel frames". In 3DXM one can choose `Parallel Frame` in the Action Menu . Now `Show Curve as Tube` illustrates the behaviour of the chosen frame. In particular the torus knots show how the parallel frames avoid "unnecessary" rotations which the Frenet frames must make.

An advantage of such parallel frames is that they neither require to assume more than *two* continuous derivatives of the curve nor that κ never vanishes—even straight lines are not exceptional curves if one works with these frames. Let $\phi(t)$ be an antiderivative of the torsion function, i.e., $\dot{\phi}(t) = \tau(t)$. Then the differential equation that determines this frame has the following simple form:

Frenet-Serret Equations for Parallel Frames:

$$\begin{aligned}
\dot{e}_1(t) &:= \quad \kappa(t)\cos(\phi(t)) \cdot e_2(t) + \kappa(t)\sin(\phi(t)) \cdot e_3(t) \\
\dot{e}_2(t) &:= -\kappa(t)\cos(\phi(t)) \cdot e_1(t) \\
\dot{e}_3(t) &:= -\kappa(t)\sin(\phi(t)) \cdot e_1(t).
\end{aligned}$$

H.K.

Free Rotational Motion of Rigid Bodies *

Part I: Angular Velocity and Rigid Motion

In this first part we will not yet consider solid objects with their inertial properties, but only so-called *rigid body kinematics*, i.e., the study of rotational motions of space. The (simpler) particle mechanics analogue of the question that we will discuss is the following: knowing the velocity curve $v(t)$ of a point how can we reconstruct the travel path $c(t)$? Since $c'(t) = v(t)$, $c(t)$ is an antiderivative of $v(t)$ and we can find it easily by integration. (Historically $v(t)$ was called the *hodograph* of the motion.)

Things to try in 3D-XplorMath

The last three entries of the Action Menu of Space Curves show demos that illustrate the present discussion. The first of these Actions, *Use Curve as Hodograph*, interpretes the space curves of 3D-XplorMath as velocity curves of a particle and reconstructs the path. The demo emphasizes that the tangent vector of the constructed path is (parallel to) the position vector of the selected space curve, the hodograph.
The second of these Actions, *Use Curve as Angular Velocity* $\vec{\omega}(t)$, reconstructs the rotational motion which has the given space curve as given angular velocity function.

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

The visualization of the motion uses a sphere with random dots and shows several consecutive points of the orbit of each random dot. One sees large orbit velocities near the equator of the rotation and small velocities near the axis of the rotation *at each moment*. – More details are explained below.
The third of these Actions, *Use Curve as Components of $\vec{\omega}(t)$ in the Moving System*, again reconstructs that rotational motion that has its angular velocity given in the moving system by the selected space curve. The space curve therefore rotates with the motion. It leaves a trace behind which shows the corresponding angular velocity curve in the observer's space. In the second Action this curve was the given one.
Finally, there is one very special space curve, *Solid Body (Euler's Polhode)*. If this space curve is selected for the third Action above then the resulting motion is *the physical motion* around the center of mass of a rigid body, taken to be a brick with edge lengths $aa \geq bb \geq cc$ and initial components of the angular momentum dd, ee, ff, see the ATO of Solid Body.

Angular Velocity given in the Observer's Space

Mathematicians and Physicists have slightly different pictures of a *motion* in their minds. A physicist sees a solid object moving in space, the movement is differentiable and all points $\vec{x}_i(t)$ of the moving object have their orbit velocities $\vec{x}_i{}'(t)$. So far these functions could also describe a mass of moving air. The word *rigid motion* means that the pairwise distances $|\vec{x}_i(t) - \vec{x}_j(t)|$ remain *constant* in time – the points $\vec{x}_i(t)$ could be the atoms of a stone. For a mathematician on the other hand the primary concept is that of a *distance preserving map of space*, and a motion is a 1-parameter family of such maps. For physicists and mathematicians it is important to understand the velocity fields $\vec{x}_i\,'(t)$ of all the particles. Physicists begin by studying rotations around *fixed* axes with *constant* angular velocities. In such a situation one can compute all the velocities $\vec{x}_i\,'(t)$ from *one* vector $\vec{\omega}$ that is parallel to the rotation axis and from the particle positions $\vec{x}_i(t)$ as follows:

$$\vec{x}_i\,'(t) = \vec{\omega} \times \vec{x}_i(t).$$

It is now a mathematical fact that differentiable families of distance preserving maps have a very similar formula for the velocities of the particles: For each time t there exists a vector $\vec{\omega}(t)$ such that we have:

$$\vec{x}_i\,'(t) = \vec{\omega}(t) \times \vec{x}_i(t).$$

And vice versa, if such a relation between the velocities and the positions holds then all pairwise distances between the particles are constant in time. Therefore mathematicians and physicists agree that a differentiable rigid motion is characterized by this relation between particle positions and particle velocities.

Now, a natural question is: If $\vec{\omega}(t)$ is a given vector function in $\mathbb{R}^3$, how can one reconstruct the rotational motion? We answer this question by constructing a so called *moving frame* $\{\vec{e}_x(t),\ \vec{e}_y(t),\ \vec{e}_z(t)\}$, a time dependent orthonormal basis. To do this we have to solve the following three ODEs:

$$\begin{aligned}
\vec{e}_x{}'(t) &= \vec{\omega}(t) \times \vec{e}_x(t), \quad \vec{e}_x(0) = (1,0,0)\\
\vec{e}_y{}'(t) &= \vec{\omega}(t) \times \vec{e}_y(t), \quad \vec{e}_y(0) = (0,1,0)\\
\vec{e}_z{}'(t) &= \vec{\omega}(t) \times \vec{e}_z(t), \quad \vec{e}_z(0) = (0,0,1).
\end{aligned}$$

Next we observe that *all* linear combinations with *constant* coefficients, i.e.
$\vec{x}(t) := x \cdot \vec{e}_x(t) + y \cdot \vec{e}_y(t) + z \cdot \vec{e}_z(t)$ satisfy
$\vec{x}\,'(t) = \vec{\omega}(t) \times \vec{x}(t)$ and are therefore orbits of the rotational motion defined by the angular velocity $\vec{\omega}(t)$.
To visualize this motion observe that for each *fixed* t the velocity field $\vec{v}(\vec{x}) := \vec{\omega}(t) \times \vec{x}$ **is** the velocity field of the ordinary rotation around the axis $\vec{\omega}(t)\mathbb{R}$ with constant angular velocity $|\vec{\omega}(t)|$.

Angular Velocity given in the Moving Space

What could it mean *to give the angular velocity of a motion in moving space*? We saw in the previous discussion that we can describe the motion of space by giving a *moving frame* $\{\vec{e}_x(t),\ \vec{e}_y(t),\ \vec{e}_z(t)\}$. The particles of moving objects have position vectors that have *constant* components a_x, a_y, a_z relative to this frame: $\vec{x}_i(t) = a_x\vec{e}_x(t) + a_y\vec{e}_y(t) + a_z\vec{e}_z(t)$. Similarly we can prescribe $\vec{\omega}(t)$ by giving its components relative to the moving frame:

$$\{\omega_x(t), \omega_y(t), \omega_z(t)\}.$$

There is again a natural question: can we again reconstruct a corresponding rotational motion for any vector function $\vec{\omega}(t)$ that is given in this way?

The answer is almost the same as for the first question, except that the three ODEs are no longer separate but are coupled by the fifth line:

$$\begin{aligned}
&\vec{e}_x{}'(t) = \vec{\omega}(t) \times \vec{e}_x(t), \quad \vec{e}_x(0) = (1,0,0)\\
&\vec{e}_y{}'(t) = \vec{\omega}(t) \times \vec{e}_y(t), \quad \vec{e}_y(0) = (0,1,0)\\
&\vec{e}_z{}'(t) = \vec{\omega}(t) \times \vec{e}_z(t), \quad \vec{e}_z(0) = (0,0,1)\\
&\text{with}\\
&\vec{\omega}(t) = \omega_x(t) \cdot \vec{e}_x(t) + \omega_y(t) \cdot \vec{e}_y(t) + \omega_z(t) \cdot \vec{e}_z(t).
\end{aligned}$$

Historical note: The given curve $\{\omega_x(t), \omega_y(t), \omega_z(t)\}$ in the moving system is called the *polhode* of the motion and the corresponding curve $\vec{\omega}(t) = \omega_x(t) \cdot \vec{e}_x(t) + \omega_y(t) \cdot \vec{e}_y(t) + \omega_z(t) \cdot \vec{e}_z(t)$ in the inertial space is called the *herpolhode*. The moving polhode and the fixed herpolhode touch each other at each time t with tangents of equal length – because the points on the momentary axis of rotation, $\vec{\omega}(t)\mathbb{R}$, have at time t the rotational velocity field $\vec{\omega}(t) \times \vec{\omega}(t) = \vec{0}$ in $\mathbb{R}^3$. A visual interpretation of this fact is that the moving polhode rolls without slipping along the fixed herpolhode. (This description actually determines the rotational motion because the origin is fixed so that the polhode has no freedom to rotate around the common tangent with the herpolhode, there is only one way to roll along.)

H.K.

Free Rotational Motion of Rigid Bodies *

What is to observe in the 3D-XplorMath exhibit
Solid Body (Euler's Polhode) ?

A brick – in the program of edge lengths $aa \geq bb \geq cc \geq 0$ – is a good example of a solid (also: rigid) body. The program illustrates the free rotational movement of such a brick (i.e. gravity is ignored):
Select *Solid Body (Euler's Polhode)*, stop the alternation between two pictures by a mouse click and select *Do Poinsot Construction From Polhode* at the bottom of the Action Menu. The resulting animation shows a freely tumbling brick. By changing aa, bb, or cc one may watch other bricks tumbling.
There are three other input parameters, dd, ee, ff. These are initial conditions for the tumbling motion. If one uses $(dd, ee, ff) \approx (1, 0.1, 0.1)$ or $(dd, ee, ff) \approx (0.1, 0.1, 1)$ then there is not much tumbling. These motions are almost rotations around the longest axis (aa) of the brick, respectively the shortest axis (cc) . The fact that these rotation axes stay close to their initial position is expressed by saying: *the rotations around the longest and the shortest axis are stable.* Now look again at the default initial conditions $(dd, ee, ff) \approx (0.1, 1, 0.1)$. One observes that the momen-

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

tary axis of rotation moves almost to the direction opposite to the initial direction and then returns back. One says: *the rotation around the middle axis of the brick is unstable.* – By putting a tape around a book and trying to throw it so that it rotates around one of the three axes one can experimentally test these theoretical predictions.

The *explanation* of this behaviour has a mathematical part and a physical part. The physical part is contained in the initial picture, the mathematical part is the connection between the initial picture and the annimation. We explain the mathematical part in

Part I: From Angular Velocity to Rotational Motion

It is available in the *Topics* part of the *Documentation.* This mathematical part has no physical limitations, any of the space curves in the program can be used as *angular velocity curve* and in the Action Menu one can select animations that show the resulting motions.
The physical part requires in addition to *angular velocity* the physical notions *tensor of inertia* and *angular momentum.* These are explained below. What can one say before this theory about the initial picture of the program? We see two space curves. The one on the sphere is the angular momentum as a function of time *in the coordinate system of the brick.* The other one is the angular velocity curve (called *Polhode*). Both are intersections of quadratic surfaces, represented by dots in the picture. The two curves

are related by a fixed linear map – given by the tensor of inertia. To emphasize this linear map the quadratic surfaces alternate between the domain and the range of this map. Finally, these two curves together determine Euler's differential equation for either of them. For example the derivative of the angular momentum curve is the cross product of the corresponding position vectors of the angular momentum curve and the angular velocity curve, in formulas: $\vec{\ell}'(t) = \vec{\ell}(t) \times \vec{\omega}(t)$. The Action Menu entry *Show Repère Mobile and ODE* illustrates this connection. The dotted curves on the sphere are solutions for other initial conditions dd, ee, ff with the same value $dd^2 + ee^2 + ff^2$. The default morph varies bb between aa and cc, it illustrates how the *family* of polhodes depends on the shape of the brick.
And here is the theory:

Part II: Tensor of Inertia and Angular Momentum

The tensor of inertia is a map that transforms *angular velocity* into *angular momentum*.
Historical note: The word *tensor* is a generic word that describes objects from linear algebra that can be given by components (indices!) with respect to a base. The *tensor of inertia* is a *linear map* from the 3-dim vector space of angular velocities to the 3-dim vector space of angular momenta. What we need below is that for each solid body there exists an orthonormal frame $\{\vec{e}_x(t),\ \vec{e}_y(t),\ \vec{e}_z(t)\}$ in the rest space of the body (i.e. moving with the body) so

that the tensor of inertia Θ is a diagonal map:

$$\begin{aligned} &\textit{angular momentum} = \Theta(\vec{\omega}(t)) = \\ &\omega_x(t) \cdot \Theta_x \vec{e}_x(t) + \omega_y(t) \cdot \Theta_y \vec{e}_y(t) + \omega_z(t) \cdot \Theta_z \vec{e}_z(t). \end{aligned}$$

$\Theta_x, \Theta_y, \Theta_z$ are called *principal moments of inertia.*

We now explain the tensor of inertia in some more detail. The result of the explanation will be the above formula for the angular momentum. We view a solid body as a collection of points of **mass** m_i and position vector $\vec{x}_i(t)$; the pairwise distances between these points are constant. The origin is the center of mass of these points, i.e. $\sum_i m_i \vec{x}_i(t) = \vec{0}$. For each mass point we have the following definitions, the corresponding notions for the solid body are obtained by summation:
linear momentum: $\vec{p}_i(t) := m_i \vec{x}_i{}'(t)$
angular momentum with respect to the origin:

$$\vec{\ell}_i(t) := \vec{x}_i(t) \times \vec{p}_i(t)$$

kinetic energy: $E_i(t) := \frac{1}{2} m_i \langle \vec{x}_i{}'(t), \vec{x}_i{}'(t) \rangle$.
The body is rigid, i.e. the distances between the points are constant, therefore there is an angular velocity function $\vec{\omega}(t)$ that relates the positions and velocities:
rotational motion: $\vec{x}_i{}'(t) = \vec{\omega}(t) \times \vec{x}_i(t)$.
angular momentum: $\vec{\ell}_i(t) = \vec{x}_i(t) \times (\vec{\omega}(t) \times \vec{x}_i(t))$.

$$=: \Theta_i(\vec{\omega}(t)).$$

This tensor of inertia is most easily understood if we use the relation between cross-product and matrix-product and insert it into the above definitions. We obtain the expressions for angular momentum and kinetic energy in terms of the tensor of inertia and the angular velocity as follows:

$$\begin{aligned}\vec{\omega}\times\vec{x} &= \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \cdot \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}\end{aligned}$$

We obtain

$$\begin{aligned}&\vec{\ell}_i(t) = \\ &m_i \begin{pmatrix} 0 & z_i & -y_i \\ -z_i & 0 & x_i \\ y_i & -x_i & 0 \end{pmatrix} \begin{pmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \\ &= m_i \begin{pmatrix} y_i^2+z_i^2 & -x_iy_i & -x_iz_i \\ -x_iy_i & y_i^2+z_i^2 & -y_iz_i \\ -x_iz_i & -y_iz_i & x_i^2+y_i^2 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \\ &= \Theta_i(\vec{\omega}) \text{ (Note the symmetry of the matrix of } \Theta_i\text{).} \\ &E_i(t) = \frac{1}{2}\langle \Theta_i(\vec{\omega}),\ \vec{\omega}\rangle.\end{aligned}$$

The symmetry of $\Theta := \sum_i \Theta_i$ implies that we have an orthonormal eigen basis for Θ. The corresponding eigen values are the principal moments of inertia, $\Theta_x, \Theta_y, \Theta_z$.

Finally, we will derive Euler's equations, a first order ODE for $\vec{\omega}(t)$. Together with part I this determines the motion of a solid body that rotates without exterior forces. We will always take the eigen basis of Θ as the moving frame of part I.

Newton's laws imply that the total angular momentum is constant in situations that are more general than the force free rotation of a solid body. We omit this general theory and show only that the conservation of angular momentum is equivalent to Euler's equations.

$$\vec{\ell}(t) := \sum_i \vec{\ell}_i(t) = \Theta(\vec{\omega}(t)) = \sum_{\xi\in\{x,y,z\}} \omega_\xi(t)\Theta_\xi \vec{e}_\xi(t)$$

implies

$$\frac{d}{dt}\vec{\ell}(t) =$$
$$\sum_{\xi\in\{x,y,z\}} \omega_\xi(t)'\Theta_\xi \vec{e}_\xi(t) + \sum_{\xi\in\{x,y,z\}} \omega_\xi(t)\Theta_\xi \vec{e}_\xi{}'(t).$$

Insert $\vec{e}_\xi{}'(t) = \vec{\omega}(t) \times \vec{e}_\xi(t)$ to get

$$\sum_{\xi\in\{x,y,z\}} \omega_\xi(t)\Theta_\xi \vec{e}_\xi{}'(t) = \vec{\omega}(t) \times \vec{\ell}(t),$$

next compute the cross product in the base given by the moving frame:

$$\vec{\omega}(t) \times \vec{\ell}(t) = \sum_{\xi\in\{x,y,z\}} \left(\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} \ell_x \\ \ell_y \\ \ell_z \end{pmatrix} \right)_\xi \cdot \vec{e}_\xi(t),$$

finally compare coefficients to get Euler's equations:

$$\begin{pmatrix} \ell_x \\ \ell_y \\ \ell_z \end{pmatrix}' = \begin{pmatrix} \Theta_x\omega_x \\ \Theta_y\omega_y \\ \Theta_z\omega_z \end{pmatrix}' = - \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \times \begin{pmatrix} \ell_x \\ \ell_y \\ \ell_z \end{pmatrix},$$

where the physics is contained in the relation between ω and ℓ :

$\ell_x = \Theta_x\omega_x,\ \ell_y = \Theta_y\omega_y,\ \ell_z = \Theta_z\omega_z.$

Considered as differential equation for the ω-components these are Euler's equations. This ODE-system implies immediately that the two quadratic functions

$$|\vec{\ell}|^2 = \ell_x^2 + \ell_y^2 + \ell_z^2 = \Theta_x^2\omega_x^2 + \Theta_y^2\omega_y^2 + \Theta_z^2\omega_z^2 \qquad \text{and}$$
$$2E = \ell_x\omega_x + \ell_y\omega_y + \ell_z\omega_z = \Theta_x\omega_x^2 + \Theta_y\omega_y^2 + \Theta_z\omega_z^2$$

are constant along solution curves. The solutions are therefore intersections of two ellipsoids. If one considers the ODE-system as differential equations for the ℓ-components then one of the ellipsoids is a sphere and the solutions $(\ell_x(t), \ell_y(t), \ell_z(t))$ are spherical curves. The choice of the ℓ-components as the functions to be determined therefore simplifies the visualization and also leads to a slightly simpler ODE-system, since the tensor of inertia enters only on the right side, linearly, into the equations.

H.K.

User Curves By Curvature And Torsion*

The exhibit allows to create examples for the standard Frenet theory of space curves. The initial dialogue allows to input user choices for curvature and torsion as functions of arc length, $\kappa(s), \tau(s)$.
The solution curves are programmed as if they were explicitly parametrized. Therefore all the Action Menu entries for parametrized curves are also available for these ODE-defined curves.

The differential equations in question are the famous

$$\begin{aligned}
&\textit{Frenet-Serret Equations:}\\
\dot{e}_1(t) &:= \qquad\qquad \kappa(t)\cdot e_2(t),\\
\dot{e}_2(t) &:= -\kappa(t)\cdot e_1(t) - \tau(t)\cdot e_3(t),\\
\dot{e}_3(t) &:= \qquad\qquad \tau(t)\cdot e_2(t).
\end{aligned}$$

For given continuous functions κ, τ these differential equations have — for given orthonormal initial values — unique orthonormal solutions $\{e_1(t), e_2(t), e_3(t)\}$.
The curve $c(t) := \int^t e_1(s)ds$ is then parametrized by arc length and has the given curvature functions κ, τ.
H.K.

* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

Userdefined Parametrized Space Curves*

These exhibits allow to input userdefined explicitly parametrized space curves in three different ways:
1.) User Cartesian: The three Cartesian coordinate functions $x(t), y(t), z(t)$ can be entered (Of course t does not have to be arc length.)
2.) User Polar: The coordinate functions can be entered in spherical polar coordinates $r(t), \theta(t), \varphi(t)$. In particular, this allows to enter spherical curves. As usual:
$x = r \cdot \sin\theta \cdot \cos\varphi,\ y = r \cdot \sin\theta \cdot \sin\varphi,\ z = r \cdot \cos\theta.$
3.) User Cylindrical: The coordinate functions can be entered in cylindrical coordinates $r(t), \theta(t), z(t)$, with the usual convention $x = r \cdot \cos\theta,\ y = r \cdot \sin\theta,\ z = z$.
Since Cylinders are isometric to the plane, this allows to create space curves that are given on all the cylinders $r = const$ by the same intrinsic geodesic curvature data $\kappa_g(s)$.
H.K.

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/