Modeling Suspension Bridge Oscillations

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Under certain physical assumptions, the Sine-Gordon Equation models the torsional oscillations of the center span of a suspension bridge. Such oscillations were observed on the Tacoma Narrows Bridge immediately prior to its spectacular failure in 1940, and were clearly responsiblefor the ensuing collapse of the span. The lattice model used for this visualization is mathematically identical to the one used for the visualization of the Discrete Sine-Gordon Lattice.in that visualization, vertical pendula replace the horizontal cross-sections of roadway transverse to the length of the span that are seen in the present visualization.

In the 3DFSdocs folder is a Quicktime movie called Tacoma Narrows Bridge.mov that shows the actual historic collapse of the Tacoma Narrows Bridge. (You can use the .Open Movie,,, selection from3D-XplorMath the File menu to view it.)

The Tacoma Narrows Bridge Failure

The Tacoma Narrows Bridge was well known for its vertical oscillations; indeed, it earned the nickname ``Galloping Gertie'' because it would oscillate vertically in winds of only 3-4 miles per hour. On November 7, 1940, just four months after it opened, as the bridge was engaging in its usual vertical activity, the nature of the motion changed suddenly, ``almost instantaneously,'' as one witness described it, from pure vertical to pure torsional. This violent twisting motion persisted for about 45 minutes, changing occasionally from one-noded to no-noded twisting, and the bridge collapsed.

We argue that the motion of suspension bridges is governed by nonlinear partial differential equations and that the inherent nonlinearity yields the fascinating behavior that was observed at Tacoma Narrows, including:

- the existence of large amplitude motion which persists over long time
- the changing nodal structure of the motion
- the dramatic change from vertical to torsional oscillation.

Large Amplitude Torsional Oscillations in Suspension Bridges

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For over fifty years, scientists in many disciplines have struggled to explain the cause of the dramatic and finally destructive oscillations of the Tacoma Narrows Bridge which preceded its collapse in 1940. We argue that the motion of suspension bridges is governed by nonlinear partial differential equations and that the inherent nonlinearity gives rise to large amplitude oscillations. Theoretical and numerical evidence to support this claim for the vertical, torsional, and traveling wave motion of suspension bridges can be found in [2] -- [8].

Recall the *linear* equation which governs the motion of a mass oscillating at the end of a spring in a damped medium with periodic forcing. If you remember your undergraduate ODE course, you know that the solution to the equation is made up of two pieces: the natural response (also called the homogeneous or complementary solution), which depends on the initial position and velocity of the mass, and the forced response (also called the particular solution), which depends on the external forcing.

If the system is damped, then the natural response decays to zero; i.e., the effect of the initial conditions ``damps away" and the long term behavior of the mass is governed entirely by the forcing. Thus, whether the initial displacement of the mass is large or small, the long term behavior will be the same.

However, this is not the case for the torsional motion of a suspension bridge. Because the equation which governs its motion is {\em nonlinear}, we find that under small periodic forcing, the solution is extremely sensitive to small changes in the initial conditions. For example, in some numerical experiments, we found that a change in the bridge's ``initial twist" of only 5 degrees could mean the difference between periodic twisting of 3 degrees and periodic twisting of over 60 degrees! [5] , [8]

Using this module, you can search for this and other types of unpredictable bridge behavior. Below I describe a model for the torsional oscillation of the main span and I suggest some experiments that you can try.

THE MODEL

We view the center span as a beam of length L suspended by cables which resist elongation according to Hooke's Law with spring constant K, but do not resist compression. Let $\theta(x,t)$ be the angular displacement at time t of the horizontal cross section of the beam located at position $x \in [0, L]$. Assuming the cables remain in tension (i.e., assuming that the cable force remains linear), θ satisfies

$$\left\{ \begin{array}{l} \theta_{tt} - \varepsilon\theta_{xx} + \delta\theta_t = -\frac{3K}{m}\sin 2\theta + f(x,t) \\ \theta(0,t) = \theta(L,t) = 0 \end{array} \right\}, \tag{1}$$

[8], where m is the mass of the cross section, δ is the damping constant, f is the external force, and ε is a physical constant related to the flexibility of the beam. The boundary condition reflects the fact that the ends of the beam are hinged.

From [1],[5], we choose

- $\delta = \varepsilon = .01$,
- $K = 1000$
- $m = 2500$

thus $\frac{3K}{m} = 1.2$.

For a cross section similar to the Tacoma Narrows bridge, wind tunnel experiments indicate that aerodynamic forces should induce approximately sinusoidal oscillations of amplitude three degrees [9], so we choose our external forcing term to be sinusoidal in time. We take

$$f(x,t) = \lambda \sin(\mu t)\rho(x)$$

where $\lambda \in [0, 0.06]$ is chosen to produce the appropriate behavior near equilibrium and the frequency μ is chosen to match the frequency of the oscillations observed at Tacoma Narrows on the day of the collapse. The frequency of the torsional motion was approximately one cycle every 4 or 5 seconds, so we take $\mu \in [1.2, 1.6]$.

The motion observed on the day of the collapse was, for the most part, a one-noded motion (i.e., no torsional displacement in the middle of the span). Occasionally, the motion changed to no-noded twisting and back again to one-noded. Thus, we choose

- $\rho(x) = 1$
- $\rho(x) = \sin\left(\frac{2\pi x}{L}\right)$ or
- $\rho(x) = \sin\left(\frac{\pi x}{L}\right)$.

To solve this PDE numerically for $t \in [0, T]$, we must discretize our domain $D = [0, L] \times [0, T]$. Let $x_0, x_1, \ldots, x_N$ form a uniform partition of the interval $[0, L]$; i.e., let the $N + 1$ points x_i be equally spaced with $0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = L$. Define $\theta_i(t) = \theta(x_i, t)$. Observe that if $\rho(x) = \sin(n\pi \frac{x}{L})$, we can express ρ independent of x by writing $\tilde{\rho}_i = \sin(n\pi \frac{i}{N})$. Suppressing the independent variable t and discretizing the θ_{xx} term via a central difference, our model (1) above becomes

$$\theta_i'' - \varepsilon \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta x)^2} + \delta\theta_i' + \frac{3K}{m}\sin 2\theta_i = \lambda \sin(\mu t)\tilde{\rho}_i \qquad (2)$$

where $\Delta x = x_{i+1} - x_i$. This system of nonlinear ODEs in θ_i is often called the discrete Sine-Gordon equation.

The current version of this module does not include the damping or forcing terms in (2); it solves the equation

$$\theta_i'' - \varepsilon \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{(\Delta x)^2} + \frac{3K}{m}\sin 2\theta_i = 0 \qquad (3)$$

Let us rewrite (3) above as

$$\theta_i'' - \left(\frac{cc}{aa}\right)^2 (\theta_{i+1} - 2\theta_i + \theta_{i-1}) + (bb)^2 \sin 2\theta_i = 0 \qquad (4)$$

so that our notation is consistent with the other modules in this category.

Illustrating Solutions to the Bridge Equation: Suggested Experiments

With the current version of this module, one can change the physical parameters and investigate the response of the "bridge". Since the length of the bridge is finite and the ends are fixed, choose "Zero

Boundary Conditions" in the Set Lattice Parameter menu under the Action Menu. Since thetorsional oscillations on the Tacoma Narrows Bridge were sinusoidal in space, choose the initial shape to be sinusoidal and select "Bridge Display"' in the Set Lattice Parameters menu.

You can select the following physical parameters:

- $(cc/aa)^2$, which is related to the flexibility of the building material. Historically, the more flexible suspension bridges such as the original Bronx-Whitestone, Golden Gate, Tacoma Narrows, and Deer Isle bridges were prone to large oscillation. Thus we suspect that smaller values of this parameter will yield large amplitude solutions.
- bb^2, which is related to 3K/m, the ratio of the cable rigidity to the mass per unit length of the roadbed. For the original Tacoma Narrows, we estimate this ratio to be about 1.2. In [8] we proved that if this parameter is "small", the equation has a unique periodic solution; in other words the behavior of the bridge is predictable. However, if this parameter is "large", the equation has multiple periodic solutions; in other words, the behavior is unpredictable. Thus, for large values of bb^2, small or large amplitude periodic oscillation might result.

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