User Defined Weierstrass Data

H. Karcher

The close connection between minimal surfaces and complex variables was worked out in the second half of the 19th century and resulted in a flourishing of the theory of minimal surfaces. One consequence of this new insight is the so-called Weierstrass representation formula for minimal surfaces. Originally this representation was a local one that only in exceptional cases allowed the representation of a complete surface. It was not until the work of Osserman (1962) that it became clear that the Weierstrass representation was in fact global. Unfortunately it is considerably more difficult to explain the global interpretation of the Weierstrass representation than it is to write down the local formula. Moreover, the input data for global numerical computations are much more complicated than what is needed in order to draw just a local piece of a surface. For these reasons, the dialog box in 3DXplorMath for user defined Weierstrass representaions only allows for the making of a local patch, and it is only the local formulation that we discuss below.

The input data for the local Weierstrass representation are two complex differentiable functions f, g defined on a region U of the complex plane. A basic fact for the representation formula to work is that the integral of a complex differentiable function along a curve gives a constant result if the curve is deformed keeping both end points fixed. We can therefore define a differentiable map F from the simply-connected region U into $\mathbf{R^3}$, by specifying a three-dimensional integrand in terms of f and g, and then integrating from a fixed base point $* \in U$ and taking

the real parts:

$$z \mapsto F(z) := \mathrm{real}\Big(\int_*^z \Big((1-g(z)^2),\ i(1+g(z)^2),\ 2g(z) \Big) f(z) dz \Big)$$

The surface piece defined by this map is always a minimal surface piece, and this formula therefore allows a user of 3DXM to view as many minimal surface pieces as desired. A short computation shows that the function g has a very nice geometric interpretation. It is customary to associate a unit vector u in $\mathbf{R^3}$ to each complex numbers w, by considering the complex plane as the x-y-plane in $\mathbf{R^3}$ and taking u to be the inverse image of w under stereographic projection. (To see examples choose in the Conformal Category for any selected function in the Action Menu *Show Image On Riemann Sphere*.) If one associates in this way a unit vector u to $w = g(z)$, then one obtains for u a unit vector orthogonal to the surface at $F(z)$. In other words: the function g composed with stereographic projection gives the normal Gauss map of the surface. The normal Gauss map is basic in the study of surfaces, and for example the various curvatures considered by differential geometers all have simple expressions in terms of the normal Gauss map. A geometric interpretation of the function f is much less immediate and this is perhaps one reason why it took so long for the above formula to be understood globally.

Sufficiently small pieces of any minimal surface are realistic models of soap films—provided the derivative of the parametrizing map F never vanishes. A point on a minimal surface where the derivative of F vanishes is called a *branch point*. Since soap films do not have branch points, one wants to look at minimal surfaces without branch points, and this can be decided from the Weierstrass data as follows:

If the function f has a zero at some point z_0 then the derivative of F at z_0 vanishes unless the other part of the integrand becomes infinite. At such points, where $g(z_0)$ is infinite, f needs to have a zero of twice the order as the infinity (pole) of g at z_0. If the order of the zero of f is larger than this then one still has a branch point and if the order of the zero is smaller, then $F(z_0)$ itself is infinite, so we do not get a point on the surface. The

default examples allow one to contemplate these facts.

Note that we use polar conformal coordinates, that is,
$z = \exp(u + iv)$, with umin $\leq u \leq$ umax, $0 \leq v \leq 2\pi$.
The unit circle is the image of $u = 0$.

The **Catenoid** data have a pole at $z = 0$, and indeed there is no corresponding point on the catenoid. The situation is similar at $z = \infty$ so that the catenoid is parametrized by a sphere minus two points. – If f is changed to $i \cdot f$ then one obtains the so called *conjugate surface*. In this case one obtains the *helicoid*, a singly periodic surface since integration of the Weierstrass integrand once around $z = 0$ adds a period to the third coordinate function. To see larger pieces of the helicoid, increase the range of v, i.e., integrate more than once around $z = 0$. – The 'Cyclic Associated Family Morph' turns the Catenoid inside out.

The **Henneberg** surface is a simple surface with branch points. These are at the 4th roots of unity since f has zeros there, but g is not infinite at those points. Note that the branch points lie on the parameter line $u = 0$, on the F-image of the unit circle. The Henneberg surface has Enneper ends at $z = 0$ and at ∞. – The default Cartesian grid touches only one branch point so that the image is less complicated than in the polar case.

The **Enneper** surfaces are parametrized by the full complex plane, and are therefore the simplest minimal surfaces to represent graphically. Note that a change of f by a unitary factor, $\exp(i\varphi) \cdot f(z)$ gives in general an isometric but non-congruent surface of the *associated family*. The Enneper surfaces are exceptional since all members of the associated family are actually *congruent*. – For the classical picture choose aa=2 and 'Cartesian Grid'. – For the most general Enneper surface take f constant and g a polynomial.

The **Trinoid**, bb = 3, has poles of f at the 3rd roots of unity. The Weierstrass map F is therefore not defined at those points. It is not obvious to see what happens at those points since the polar coordinates that the user defined surfaces use are not adapted to the situation. Compare the case ee = 3 of the Sym-

metric kNoids in the list of 3DXM minimal surfaces. – Choosing the default 'Cartesian Grid' is most instructive for the Fournoid, our bb = 4 default. As in all cases the 'Cyclic Associated Family Morph' is interesting.

Even the simplest surfaces can be given by Weierstrass data from which one does not immediately recognize the surface. Try:
$g(z) = (z^k - 1)/(z^k + 1), \quad f(z) = 0.8\mathbf{i}(z^k + 1)^2,$
$-1.8 \leq u \leq 0.2,\ 0 \leq v \leq 2\pi, \quad (k = 0, 1, 2, 3, 4).$