About Ward Solitons

The *Ward Equation*, also called the *modified* $2+1$ *chiral model*, is the an equation for a map $J : \mathbf{R}^{2,1} \to \mathbf{SU}(n)$, namely:

$$(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y - [J^{-1}J_t, J^{-1}J_y] = 0.$$

This non-linear wave equation is obtained by a dimension reduction and a gauge fixing of the self-dual Yang-Mills equation on $\mathbf{R}^{2,2}$. It is an equation for an integrable system, and as such it has a Lax pair. We explain this next.

Let

$$u = \frac{1}{2}(t+y), \quad v = \frac{1}{2}(t-y)$$

be the light-cone coordinate system for the yt-plane, and given smooth maps A and B from $\mathbf{R}^{2,1}$ to $su(n)$, we consider the following linear system for a map $\psi : \mathbf{R}^{2,1} \times \mathbf{C} \to GL(n, \mathbf{C})$:

$$\begin{aligned}(\lambda\partial_x - \partial_u)\psi &= A\psi \\ (\lambda\partial_v - \partial_x)\psi &= B\psi.\end{aligned}$$

R. Ward showed that if $\psi(x,u,v,\lambda)$ is a solution of $(*)$ and satisfies the $\mathbf{U}(n)$-reality condition

$$\psi(x,u,v,\bar{\lambda})^*\psi(x,u,v,\lambda) = I,$$

then $J(x,u,v) = \psi(x,u,v,o)^{-1}$ is a solution of the Ward equation.

We call ψ an *extended solution* of J. The *degree* of ψ is defined to be the number of poles for λ, counted with multiplicities. A solution J of the Ward equation is called a *Ward soliton* if J has an extended solution $\psi(x,u,v,\lambda)$ that is rational in λ, satisfies $\psi(x,u,v,\infty) = I$, and if for each t, J is asymptotically constant as (x,y) approaches ∞. A Ward soliton is called a *k-soliton* if the smallest degree of extended solutions of J is k.

Ward used the solution of the Riemann-Hilbert problem to write down all solitons whose extended solutions have only simple poles, and used a limiting method to construct 2-solitons whose extended solution have one double pole. He showed that these 2-solitons have non-trivial scattering. Anand and Ioannidou-Zakrzewski found Ward solitons whose extended solutions have only one pole of multiplicities 2 and 3 (see the Anand-Ward Solitons submenu of the Surfaces Catergory). Dai and Terng used Bäcklund transformations and an order k limiting method to construct all Ward solitons. Their method gives explicit formula for Ward solitons. In fact, given constant $z_1, \ldots, z_r \in \mathbf{C} \setminus R$, positive integers $n_1, \ldots, n_r$, and rational functions $v_0, \ldots, v_k$, they wrote down a k-soliton whose extended solution has poles at $\lambda = z_1, \ldots, z_r$ with multiplicities $n_1, \ldots, n_r$ respectively. We call

$$(z_1, \ldots, z_r, n_1, \ldots, n_r)$$

the *pole data* of the Ward soliton. For detail of Dai and Terng's construction, we refer the reader to their paper "Bäcklund transformations, Ward solitons, and unitons". This algorithm was used to write the code for the Ward solitons in the 3D-XplorMath. The program shows the wave profile of the energy density

$$E(x, y, t) = ||J^{-1}J_x||^2 + ||J^{-1}J_y||^2 + ||J^{-1}J_t||^2.$$

The default Morph shows a sequence of these profiles for an increasing time t_i for the $SU(2)$ Ward solitons.

A Ward soliton with pole data $(z_1, \ldots, z_r, n_1, \ldots, n_r)$ represents the interaction of r solitons with pole data (z_1, n_1), ..., (z_r, n_r) respectively. The shapes of these r solitons are preserved after the interaction, but with possible phase shift. However, the solitons with pole data (z, k) do not share this common phenomenon of soliton equations. In fact, the localized lumps scatter after interaction.

To use the program to see the wave profiles of the $SU(2)$ Ward solitons, you should be in the Surface Catergory and make a selection from the Ward Solitons submenu of the Surfaces menu. The submenu MultiSoliton$_I$ refers to Ward k-solitons whose extended solution has a multiplicity k pole at $\mathbf{i}= \sqrt{-1}$. Similarly, 3-soliton$_{IIZ}$ means Ward 3-soliton whose extended solution has poles $\mathbf{i}$, $\mathbf{i}$, Z, etc. After choosing the Ward Solitons submenu, you can then specify the pole locations and rational maps from the submenu Ward Soliton Settings to construct the corresponding Ward solitons. For simplicity of writing the codes, we only wrote code for k-solitons with $1 \leq k \leq 4$ and the rational maps from $\mathbf{C}$ to $\mathbf{C}^2$ are of the form $(1, a_i(w))$, where $a_i(w)$ is a polynomial in w of degree less than 5. You may specify the coefficients of these a_i's in the Ward Soliton Settings. Since the maximum of the energy density of these solitons is often large, you may choose a scaling factor in the Settings menu. If you choose Energy Scale Factor := 0.05, this means the wave profile you see from the program is for $0.05E(x, y, t)$.

To see some Quicktime movies made using the program, click on the following link:

http://3D-XplorMath.org/WardSolitons/

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