Willmore minimizing $\mathbb{RP}^2$s (Bryant)

For any surface immersion $X : M^2 \to \mathbb{R}^3$, the non-negative quantity $W(X) = \int_M (H^2 - K)\,|\mathrm{d}A|$ is the *Willmore energy* of the immersion X. It is a conformal invariant since the integrand is a conformally invariant density.

By a theorem of Li and Yau, when M is compact,

$$W(X) \geq 4\pi\mu_X - 2\pi\chi(M)$$

where $\chi(M)$ is the Euler characteristic of M and μ_X is the maximum cardinality of $X^{-1}(y)$ for $y \in \mathbb{R}^3$.

When $M = \mathbb{RP}^2$, one has $\chi(M) = 1$ and, by a theorem of Banchoff, $\mu_X \geq 3$. Thus, $W(X) \geq 10\pi$. In 1984, R. Bryant classified the Willmore minimizing immersions for $\mathbb{RP}^2$, i.e., those with $W(X) = 10\pi$. He showed that, up to conformal transformations and reparametrizations, there is a non-compact 2-parameter family of minimizers, and that it can be compactified by adding in a branched immersion.

One starts with a family of meromorphic null space curves $f_{p,q}$ defined as follows: Let $(p, q) \in \mathbb{C}^2$

satisfy $|p|^2{+}|q|^2{=}1$. Define the polynomial

$$P_{p,q}(z) = 2q\,z^6 + 3p\,z^5 + 2\sqrt{5}\,z^3 + 3\bar{p}\,z - 2\bar{q}$$

and then the meromorphic null curve

$$f_{p,q}(z) = \frac{1}{P_{p,q}(z)} \begin{pmatrix} \mathrm{i}\,(2\bar{p}\,z^6 - 3\bar{q}\,z^5 + 3q\,z + 2p) \\ (2\bar{p}\,z^6 - 3\bar{q}\,z^5 - 3q\,z - 2p) \\ -\mathrm{i}\,(2q\,z^6 + 3p\,z^5 - 3\bar{p}\,z + 2\bar{q}) \end{pmatrix}.$$

The mapping $f_{p,q}$ has poles at the roots of $P_{p,q}$ (plus $z = \infty$ if $q = 0$). These poles are six in total number and are distinct unless $\sqrt{5}\,p^3{+}8\,q^2 = 0$ holds, in which case, $f_{p,q}$ has two triple poles. It satisfies

$$\begin{aligned} f_{p,q}(z) &= \overline{f_{p,q}(-\bar{z}^{-1})}, \\ f_{\lambda^2 p,\lambda^3 q}(\lambda^{-1} z) &= R(\lambda^5)\, f_{p,q}(z), \\ f_{p,q}(z) &= S\,\overline{f_{\bar{p},\bar{q}}(\bar{z})} \end{aligned}$$

when $\lambda = \cos\theta + \mathrm{i}\,\sin\theta \in S^1$ and where

$$R(\lambda) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Recall that $\mathbb{RP}^2$ is the quotient of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ by the antiholomorphic involution $z \mapsto -\bar{z}^{-1}$ and set

$$X_{p,q}(z) = \mathrm{Re}\big(f_{p,q}(z)\big) = \frac{1}{2}\left(f(z) + \overline{f(z)}\right).$$

Then $X_{p,q}$ satisfies

$$X_{p,q}(z) = X_{p,q}(-\bar{z}^{-1}).$$

Since $f_{p,q}$ is a meromorphic null immersion of $\mathbb{CP}^1$, it follows that $X_{p,q}$ is a minimial conformal immersion into $\mathbb{R}^3$ of $\mathbb{RP}^2$ minus the set D that is the image of the set of zeros of $P_{p,q}$ (which is therefore either three points in $\mathbb{RP}^2$ or a single point). Moreover, in the case that the roots of $P_{p,q}$ are distinct, $X_{p,q}$ has 3 planar ends of zero logarithmic growth.

Thus, when one embeds $\mathbb{R}^3$ into $S^3 = \mathbb{R}^3 \cup \{\infty\}$ conformally, $X_{p,q}$ extends across the three 'missing' points of $\mathbb{RP}^2$ to become a conformal Willmore immersion

$$X_{p,q} : \mathbb{RP}^2 \to S^3$$

The Willmore functional on these immersions is equal to 10π since the degree of the Gauss map of $X_{p,q}$ is 5. In fact, the Gauss map is represented by

$$g_{p,q}(z) = \frac{z^2\big(z^3 + \sqrt{5}(\bar{p}\,z - \bar{q})\big)}{1 + \sqrt{5}(p\,z^2 + q\,z^3)},$$

so that the Willmore energy density is given by

$$-K\,|\mathrm{d}A| = \frac{4\left|g'_{p,q}(z)\right|^2 |\mathrm{d}z|^2}{\left(1 + \left|g_{p,q}(z)\right|^2\right)^2}.$$

Bryant showed that every Willmore minimizing immersion of $\mathbb{RP}^2$ is conformally equivalent to some $X_{p,q}$ with $\sqrt{5}\,p^3 + 8\,q^2 \neq 0$. Also, $X_{p,q}$ is conformally equivalent to $X_{p',q'}$ if and only if (p',q') is equivalent to (p,q) under the action generated by the involution $(p,q) \mapsto (\bar{p},\bar{q})$ and the circle action

$$\mathrm{e}^{\mathrm{i}\theta} \cdot (p,q) = \big(\mathrm{e}^{2\mathrm{i}\theta}p, \mathrm{e}^{3\mathrm{i}\theta}q\big).$$

(The case $(p,q) = (0,1)$ was independently discovered by Rob Kusner.) The locus $\sqrt{5}\,p^3 + 8\,q^2 = 0$ is a single orbit under this action, so after removing this singular orbit, the quotient space can be thought of as a closed 2-disk with a boundary point removed.

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