Boy Surfaces,

following **Apery** and **Bryant-Kusner** *

See Möbius Strip and Cross-Cap first.

All the early images of the projective plane in $\mathbb{R}^3$ had singularities, the **Boy Surface** was the first *immersion*. Since the projective plane is *non-orientable*, no embedding into $\mathbb{R}^3$ exists and *self-intersection curves* have to occur on the image. In fact, the self-intersection curve of the Boy surface is also *not* embedded, the surface has a *triple point*. Boy discovered the surface while working for his PhD under Hilbert. Boy's construction was differential topology work, his surface has no special local geometry.

Apery found an **Algebraic Boy Surface**. Moreover, his surface is covered by a 1-parameter family of ellipses. This is his *Parametrization*:

$$F_{Apery}(u,v) = \left(\begin{array}{c} \frac{\cos^2(u)\cos(2v)+\sin(2u)\cos(v)/\sqrt{2}}{\sqrt{2}-\sin(2u)\sin(3v)} \\ \frac{\cos^2(u)\sin(2v)-\sin(2u)\sin(v)/\sqrt{2}}{\sqrt{2}-\sin(2u)\sin(3v)} \\ \frac{\sqrt{2}\cos^2(u)}{(\sqrt{2}-\sin(2u)\sin(3v))} \end{array} \right).$$

This parametrization is obtained by restricting the following *even* map from $\mathbb{R}^3$ to $\mathbb{R}^3$ to the unit sphere:

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

$$
\begin{aligned}
denom &:= (\sqrt{2} - 6xz)(x^2 + y^2) + 8x^3 z \\
F_x &:= ((x^2 - y^2)z^2 + \sqrt{2}xz(x^2 + y^2))/denom \\
F_y &:= (2xyz^2 - \sqrt{2}yz(x^2 + y^2))/denom \\
F_z &:= z^2(x^2 + y^2)/denom
\end{aligned}
$$

The image of the unit sphere is also an *image of the projective plane* since $(F_x, F_y, F_z)(-p) = (F_x, F_y, F_z)(p)$.

The `Default Morph` starts with a band around the equator, which is a Möbius Strip with *three* halftwists. The complete surface is obtained by attaching a disk (centered at the ploar center). 3DXM supplies a second morph, `Range Morph` in the Animation Menu. It starts with a band around a meridian, which is another Möbius Strip with *one* halftwist. This Möbius Strip is moved over all the meridians, covering the surface with embedded Möbius Strips.

Bryant-Kusner Boy Surfaces are obtained by an inversion from a minimal surface in $\mathbb{R}^3$. The minimal surface is an immersion of $\mathbb{S}^2 - \{6 \text{ points}\}$ such that antipodal points have the same image in $\mathbb{R}^3$, so that the minimal surface is an image of the projective plane minus three points. The six punctures are three antipodal pairs, and the minimal surface has so called *planar ends* at these punctures. This is the same as saying that the Weierstrass-integrand has no residues, hence can be explicitly integrated. In this context it is important that the inversion of a planar end has a puncture that can be *smoothly* closed by adding one point.

The closing of the three pairs of antipodal ends thus gives a triple point on the smoothly immersed surface which is obtained by inverting the minimal surface.

As `Default Morph` and as `Range Morph` we took the same deformations as in the algebraic case. The first emphasizes the equator Möbius Strip with *three* halftwists, the second covers the surface with embedded Möbius Strips that have meridians as center lines.

A *Parametrization* is obtained by first describing the minimal surface as an image of the Gaussian plane, then invert it in the unit sphere. Parameter lines come by taking polar coordinates in the unit disk.

$MinSurf(z) := \mathrm{Re}\,(V(z)/a(z)) + (0, 0, 1/2)$, where
$a(z) := \left(z^3 - z^{-3} + \sqrt{5}\right)$ and
$V(z) := \left(i(z^2 + z^{-2}),\ z^2 + z^{-2},\ \frac{2i}{3}(z^3 + z^{-3})\right)$.
Finally the inversion:

$$Boy(z) := \frac{MinSurf(z)}{||MinSurf(z)||^2}.$$

H.K.