Surfaces
in 3D-XplorMath, a Visualization Program

Explicit versus Implicit Surfaces
Curvature Properties of Surfaces

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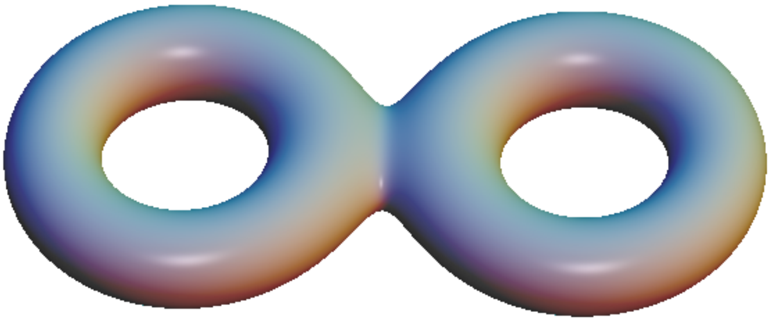
Explicit versus Implicit Surfaces,
in particular Level Sets of Functions *



Surfaces in $\mathbb{R}^3$ are either described as *parametrized images* $F : D^2 \to \mathbb{R}^3$ or as *implicit surfaces*, i.e., as levels of functions $f : \mathbb{R}^3 \to \mathbb{R}$, as the set of points where f has a fixed value v: $\{x \in \mathbb{R}^3;\ f(x) = v\}$. *Graphs* of functions $h : \mathbb{R}^2 \mapsto \mathbb{R}$ are both: $F(u, v) := (u, v, h(u, v))$ is a *parametrization* and $f(x, y, z) := h(x, y) - z$ is a level function, $f = 0$ the *implicit equation*.
For most simple surfaces one has both representations, examples in 3DXM: *All Quadratic Surfaces, Tori, Cyclides, Cross-Cap, Steiner Surface, Algebraic Boy Surface, Whitney Umbrella.* In each case, the explicit and the implicit version open the same `Documentation`.
One can more easily make images of parametrized surfaces than of implicit surfaces, because every point $p \in D$ can be mapped with the given function F to obtain 'explicitly' a point $F(p)$ of the surface. Note however that the opposite problem: "Given a point in $\mathbb{R}^3$, decide whether it lies on

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the surface" does not have an easy answer. For an implicit surface, on the other hand, it is easy to decide whether a given point in $\mathbb{R}^3$ is on the surface (simply check $f(x) = v$), but no point is given explicitly, one has to use some algorithm to find points $x \in \mathbb{R}^3$ which satisfy $f(x) = v$.
And even after one has found many points on the surface, how does one connect them, what is a good way to represent the surface? The method of *raytracing* has been invented as one solution. Choose some center point C, think of it to be near the eyes of the viewer. Connect each pixel of the screen with C by a line and decide whether this line meets the surface. Of all the intersection points on the line choose the one closest to C, compute the normal of the surface at this point x (i.e. compute grad $f(x)$) and decide with this information what light (from fixed light sources) will be reflected by the surface at x towards C. Color that pixel accordingly. In this way one produces an image which presents the surface as if it were an illuminated object. The computation used to take very long, but todays computers do such pictures while you wait, but not quite fast enough for real time rotations. These pictures look very realistic, but of course they show only what is visible from the viewer.
In 3D-XplorMath a second method is offered. Imagine that the surface is intersected with random lines until around 10.000 points have been found on the implicit surface. Then red-green stereo is used to project these points to the screen. When viewing through stereo glasses we see all

these points in the correct position in space and our brain interpolates them and lets us see a surface in space. This representation shows all parts of the surface (within some viewing sphere), not just the front most portions. Since one can achieve fairly uniform distributions of points on level surfaces, one sees many points in the direction towards *contours* of the surface. This emphasis of the contour points is so strong that one gets a fair impression of the surface even if one does not look through red-green glases. This method is fast enough for real time rotations.
Once an implicit surface has been drawn, one has solved the problem of computing the 3D-data of surface points selected by mouse on the screen. One can therefore more easily move geometric attributes, like curvature circles, around on an implicit surface than on a parametrized surface. See in both cases the Action Menu entry `Move Principal Curvature Circles.`

What surfaces can one see?

In addition to the simple surfaces already mentioned we have two groups. *Alebraic Surfaces* which have been studied because of their *singularities*, these have established names and extensive literature. And *Compact Surfaces of higher genus*, these are added because such surfaces do not come with explicit parametrizations. (Their names are given in 3DXM and not known elsewhere.) Already fairly simple functions may have level surfaces which are more complicated than tori, they are called bretzel surfaces of genus $g > 1$.

How to find functions with compact levels of genus ≥ 2. As an example, consider *two circles* of radius $r = 1$, in the x-y-plane, with midpoints $\pm cc$ on the x-axis. These two circles are described as the intersection of the x-y-plane: $\{g(x,y,z) := z = 0\}$ with the zero set of the function

$$h(x,y,z) := \frac{((x-cc)^2 + y^2 - 1) \cdot ((x+cc)^2 + y^2 - 1)}{1 + (1+cc)(x^2+y^2)}.$$

The denominator prevents the function from growing too fast, the weight factor $1 + cc$ is experimental. Next define

$$f(x,y,z) := h(x,y,z)^2 + (1+cc)g(x,y,z)^2.$$

Clearly, the zero set of f is the union of the two circles, which is *not* a surface, because grad f vanishes along this zero set. However, most of the levels $\{(x,y,z); f(x,y,z) = v > 0\}$ are surfaces without singularities. If the two circles intersect ($0 < cc < 1$), then for small $v = ff$ the levels are the boundary of a thickening of the two circles, i.e., surfaces of genus 3. As ff increases either the middle hole or the two outside holes close first (depending on cc). For large ff the level surfaces are (not completely round) spheres. Each time such a topological change occurs we observe one special surface, it is not smooth like the other levels, but has one or more cone like singularities.
If $cc > 1$ then, for small ff, the levels are disjoint tori. As ff increases, either the tori grow together to a genus 2 surface, or the holes of the tori close first and later the two sphere-like surfaces grow together.

This family is called *Pretzel* in 3DXM.

Functions with compact levels in 3D-XplorMath

One should always experiment with the level value v of the function f. In 3DXM: $v = ff$. For small values of ff one will see how the function was designed by guessing the degenerate level $f = 0$. The `Default Morph` often varies ff, for example showing non-singular levels converging to the singular one. In some cases other parameters are morphed, for example to get larger values of the genus g. Some cases offer: `Flow to Minimum Set` $\{f = 0\}$ (see Action Menu). (Artificial looking denominators in the following prevent the function f from growing too fast.)

Bretzel 2, a genus 2 tube around a figure 8:

$$f(x,y,z) := \frac{\big(((1-x^2)x^2 - y^2)^2 + z^2/2\big)}{(1 + bb(x^2+y^2+z^2))}.$$

Bretzel 5, a genus 5 tube around two intersecting ellipses:
$f(x,y,z) := ((x^2 + y^2/4 - 1)\cdot(x^2/4 + y^2 - 1))^2 + z^2/2.$

Pilz, a genus 3 tube around circle and orthogonal ellipse:
$f(x,y,z) :=$
$((x^2+y^2-1)^2+(z-0.5)^2)\cdot(y^2/aa^2+(z+cc)^2-1)^2+x^2)$
$-\,dd^2(1+bb(z-0.5)^2)$. `Default Morph:` $0.03 \le cc \le 0.83.$

Orthocircles, a genus 5 tube around three intersecting orthogonal circles ($aa = 1,\ ff = 0.05$) or a tube around three Borromean ellipses ($aa = 2.3$, $ff = 0.2$):
$f(x,y,z) :=$
$((x^2/aa + y^2 - 1)^2 + z^2)\cdot((y^2/aa + z^2 - 1)^2 + x^2)\cdot$
$((z^2/aa + x^2 - 1)^2 + y^2).$
Use: `Flow to Minimum Set` $\{f = 0\}$ (from Action Menu).

DecoCube, tube around six circles of radius cc on the faces of a cube. Genus 5,13,17, depending on cc, ff:
$f(x,y,z) := ((x^2+y^2-cc^2)^2+(z^2-1)^2)\cdot$
$((y^2+z^2-cc^2)^2+(x^2-1)^2)\cdot((z^2+x^2-cc^2)^2+(y^2-1)^2)$.
`Default Morph:` $ff = 0.02,\ 0.25 \le cc \le 1.3$.
Use: `Flow to Minimum Set` $\{f=0\}$ (from Action Menu).

DecoTetrahedron has as its minimum set four circles on the faces of a tetrahedron. The formula is similar but more complicated than the previous one. cc changes the radius of the circles, bb changes their distance from the origin, ff selects the level. Use: `Flow to Minimum Set` to see the circles used for the current image. The `Default Morph` changes cc and with it the genus.

JoinTwoTori is a genus 2 surface such that the connection between the two tori does not much distort them if ff is small. (It is used for genus-2-knots in Space Curves.)

$$\begin{aligned} Tor_{right} &:= ((x-cc)^2+y^2+z^2-aa^2-bb^2)^2 \\ &\qquad + 4aa^2(z^2-bb^2) \\ Tor_{left} &:= ((x+cc)^2+y^2+z^2-aa^2-bb^2)^2 \\ &\qquad + 4aa^2(z^2-bb^2) \end{aligned}$$

$$f(x,y,z) := \frac{Tor_{right}\cdot Tor_{left}}{1+(x-cc)^2+(x+cc)^2+y^2+z^2/2}.$$

The `Default Morph:` $0.01 \le ff \le 2.5$ joins the tori.

Algebraic Functions with Singularities in 3DXM

Cayley Cubic :

$f(x, y, z) := 4(x^2 + y^2 + z^2) + 16\, x\, y\, z - 1, \quad ff = 0.$

This cubic has 4 cone singularities at the vertices of a tetrahedron. The other surfaces in the `Default` ff`-Morph` are nonsingular.

Clebsch Cubic :

$f(x, y, z) :=$

$81(x^3 + y^3 + z^3) - 189(x^2(y + z) + y^2(z + x) + z^2(x + y)) + 54xyz + 126(xy + yz + zx) - 9(x^2 + x + y^2 + y + z^2 + z) + 1.$

This cubic has no singularities but is famous for the 27 lines that lie on it. The lines are shown in 3DXM. The surface has tetrahedral symmetry.

Doubly Pinched Cubic :

$f(x, y, z) := z(x^2 + y^2) - x^2 + y^2.$

This cubic has two pinch-point singularities at ± 1 on the z-axis. The segment between the singularities lies on it. The whole z-axis satisfies the equation; the `Default Morph` shows how an infinite spike converges to this line.

Kummer Quartic :

$\lambda := (3aa^2 - 1)/(3 - aa^2),$

$$f(x, y, z) := (x^2 + y^2 + z^2 - aa^2)^2 - \lambda((1 - z)^2 - 2x^2)((1 + z)^2 - 2y^2), \quad aa = 1.3.$$

This quartic has 4+12 cone singularities and tetrahedral symmetry. Six noncompact pieces, each with two cone points, are connected by five compact pieces which look like curved tetrahedra. The singularities survive small changes, see the `Default Morph` : $1.05 \le aa \le 1.5, ff = 0.$

Barth Sextic :
$c_1 := (3+\sqrt{5})/2,\ c_2 := 2+\sqrt{5}$
$f(x,y,z) :=$
$4(c_1x^2-y^2)(c_1y^2-z^2)(c_1z^2-x^2)-c_2(x^2+y^2+z^2-1)^2.$
Barth's Sextic has icosahedral symmetry. 20 tetrahedron-like compact pieces are placed over the vertices of a dodecahedron so that each tetrahedron has 3 of its vertices at midpoints of dodecahedron edges. This accounts for 30 of the cone singularities. Each of the 20 outward pointing vertices of the tetrahedra is connected via a cone singularity to a cone-like noncompact piece of the Sextic. The `Default Morph` embeds this singular surface in a family of nonsingular sextics. `Use Raytrace Rendering`.

D4 :
$f(x,y,z) := 4x^3 + (aa-3x)(x^2+y^2) + bbz^2$
This family of cubics has a $D4$-singularity. At $bb = 0$ the family degenerates into three planes, intersecting along the z-axis.

UserDefined : Our example is the *Cayley Cubic*, see above.

H.K.

Curvature Properties of Surfaces *

Any curvature discussion of surfaces assumes some knowledge about curvature properties of curves.

Planar Curves have, at each point $c(s)$, only one kind of curvature. Consider a circle through $c(s)$ that has the same first and second derivative as the curve at $c(s)$. Such a circle is called *osculating circle*, it approximates the curve better than any other circle and it can easily be recognized if $c'''(s) \neq 0$: The circle has the same tangent as the curve and is on different sides of the curve before and after $c(s)$. See `Osculating Circles` in the Action Menu. The radius of this circle is called *curvature radius* $r(s)$, and the *curvature* is defined as $\kappa(s) := 1/r(s)$. If s is the arclength parameter, i.e. $|c'(s)| = 1$, then $\kappa(s) = |c''(s)|$. The *fundamental theorem* for planar curves states:

> If a continuous curvature function $\kappa(s)$ is given then there exists, up to congruence, exactly one planar curve with this curvature function.

Space Curves have the same definition of *osculating circle* and of *curvature* $\kappa(s)$ as the plane curves. If s is arclength on c then $c''(s)$ is called *principal curvature vector* and $\vec{h}(s) = c''(s)/|c''(s)|$ is called *principal normal* .

In addition, space curves have a second kind of curvature, the *rotation speed* of the principal normal, also called *torsion* $\tau(s) := |\vec{h}'(s)|$. The *fundamental theorem* for space

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curves (roughly) states:

> If continuous curvature functions $\kappa(s), \tau(s)$ are given then there exists, up to congruence, exactly one space curve with these curvature functions.

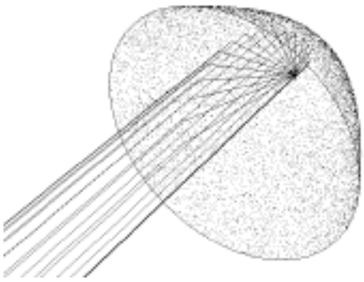
Surfaces have as their most visible curvature properties their *normal curvatures*: Consider at a surface point p the intersection of the surface with all its normal planes at p. The curvatures of these *normal sections* are the normal curvatures. They can be computed as follows: Let N be a unit normal field along the surface and $c(s), c(0) = p$ a normal section with $c'(0) = \vec{e}, |\vec{e}| = 1$. Then its curvature, the normal curvature in the direction $\vec{e}$, is

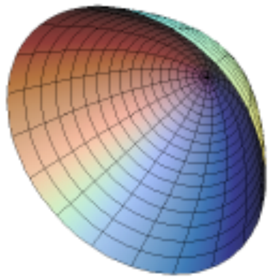
$$\kappa(p, \vec{e}) = |c''(0)| = \langle D_{\vec{e}} N, \vec{e} \rangle.$$

These normal curvatures have a minimum and a maximum, called the principal curvatures κ_1, κ_2 at p. The corresponding vectors $\vec{e}$ are called the *principal directions* $\vec{e}_1 \perp \vec{e}_2$. $H := \kappa_1 + \kappa_2$ and $K := \kappa_1 \cdot \kappa_2$ are *mean curvature* and *Gauss curvature*.
If the surface is given by an *explicit parametrization*, it is straight forward to compute these data. If the surface is given by an *implicit equation* $f(x, y, z) = 0$, one chooses $N(x, y, z) := \operatorname{grad} f / |\operatorname{grad} f|$ and computes $\langle D_{\vec{e}} N, \vec{e} \rangle$, as before.
For these surfaces one finds in the Action Menu the entries:
`Add Principal Curvature Fields`,
`Move Principal Curvature Circles`.
They allow to view and move the above curvature objects.

The Paraboloid *

See in Documentation: About Quadratic Surfaces.





The Paraboloid in 3D-XplorMath is *parametrized* as

$$x = aa \cdot u \cdot \cos(v), \quad y = bb \cdot u \cdot \sin(v), \quad z = cc \cdot u^2 - dd,$$

with the default $aa = bb = 1,\ cc = 0.4,\ dd = 2$. It is given *implicitly* by $f(x, y, z) := \frac{(z+dd)}{cc} - (\frac{x}{aa})^2 - (\frac{y}{bb})^2 = 0$.

The paraboloid is shown together with a few rays parallel to the z-axis, the axis of revolution symmetry of this surface. These rays are reflected in the surface and continued until they meet in the focal point of this paraboloid. This image looks somewhat like the reflector of a car headlight together with the rays from the light bulb, reflected into parallel rays. The `default Morph` varies cc so that the image changes from a headlight reflector to a satellite antenna, with incoming parallel rays concentrated on the receiver at the focal point of the antenna.

The entry `Remove Focal Rays` in the Action Menu returns to the standard rendering for surfaces. Only in `Wireframe`

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`Display` can one switch on the focal rays in the Action Menu.

For geometric arguments concerning the focal point see: `Parabola` in the Plane Curve category.

H.K.

The Ellipsoid *

See in Documentation: About Quadratic Surfaces.

The Ellipsoid in 3D-XplorMath is *parametrized* as

$$x = aa \cdot \sin u \cdot \cos v, \quad y = bb \cdot \sin u \cdot \sin v, \quad z = cc \cdot \cos u,$$

with the default $aa = bb = 1.5,\ cc = 2.0$. It is given by the

Implicit Equation

$$f(x, y, z) := (x/aa)^2 + (y/bb)^2 + (z/cc)^2 = 1.$$

In 3D-XplorMath the Ellipsoid is shown together with a few rays which leave one focal point, are reflected in the surface and come together again in the other focal point. This illustrates the use of the Ellipsoid as a *Whispering Gallery*. A whispering gallery may be realized by an Ellipsoid ceiling in a pub so that the conversations at one table can be heard at another table. Whispering galleries were also built in royal parks with some ellipsoid reflector near a table for visitors placed at one focal point and a hidden chair for the listener at the other focal point.

The first `default Morph` varies the size of the Ellipsoid. One can also select in the View Menu `Morph Light Source Of Rays` to illustrate that the rays do not come together at one point unless they start from a focal point.

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By selecting `Remove Focal Rays` in the Action Menu one returns to the standard rendering of surfaces. One may turn on the focal rays only if `Wireframe Display` is selected.

For geometric arguments concerning the focal points see: `Ellipse` in the Plane Curve category.

H.K.

About Quadratic Surfaces *

Quadratic surfaces in $\mathbb{R}^3$ are the solution sets of quadratic equations (see also: `About Implicit Surfaces`)

$$h(x,y,z) := Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz$$
$$+ Gx + Hy + Jz + K = 0.$$

Explicit parametrizations are given at the end.
There are poor examples, i.e.

with no solutions: $x^2 + y^2 + z^2 + 1 = 0$,

solutions consisting of a point: $x^2 + y^2 + z^2 = 0$,

or solution sets consisting of a line: $x^2 + y^2 = 0$,

or solution sets like that of a linear function: $x^2 = 0$.

But under mild assumptions, namely that the derivative of h does not vanish on (most of) the solution set, we get more interesting surfaces (possibly with singularities) as solution sets.
For products of linear functions we have intersections of (or parallel) planes: $(x - y + a)(x \pm y + b) = 0$,
we may have cylinders over quadratic curves, e.g. if the equation does not contain z:

elliptic cylinder $x^2 + y^2 - 1 = 0$,

hyperbolic cylinder $x^2 - y^2 - 1 = 0$,

parabolic cylinder $x^2 - y = 0$.

There are various cones, namely solution sets which con-

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tain for each solution $(x, y, z) \neq (0, 0, 0)$ the whole line $r(x, y, z), r \in \mathbb{R}$ through (in this example) 0. For example, if we intersect the previous cylinders with the plane $z = 1$ and take the cone with vertex at $(0, 0, 0)$ then these are described by the following (so called "homogenous") equations: $x^2 + y^2 - z^2 = 0$, $x^2 - y^2 - z^2 = 0$, $x^2 - yz = 0$.
And finally we have the quadratic surfaces which have neither singular points nor are they cylinders:

Ellipsoids	$x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0,$
1-sheeted Hyperboloids	$x^2/a^2 + y^2/b^2 - z^2/c^2 - 1 = 0,$
2-sheeted Hyperboloids	$x^2/a^2 + y^2/b^2 - z^2/c^2 + 1 = 0,$
Elliptic Paraboloids	$x^2/a^2 + y^2/b^2 - z = 0,$
Hyperbolic Paraboloids	$x^2/a^2 - y^2/b^2 - z = 0.$

All other quadratic surfaces are obtained via coordinate transformations from these examples. Try the
Experiment: Select Implicit from the Surface Menu and type any quadratic equation into UserDefined. Compare the displayed surface with those described above. (Note that there may be no solutions.)

The 1-sheeted hyperboloids and the hyperbolic paraboloids have an unexpected special property, they carry two families of straight lines, see also: `About Ruled Surfaces`.
The *hyperbolic paraboloid:* $x^2 - y^2 - z = 0$ is cut by the parallel family $x + y = \mathit{const}$ of planes in (disjoint) lines and also by the parallel planes $x - y = \mathit{const}$.
The *1-sheeted hyperboloid:* $x^2 + y^2 - z^2 - 1 = 0$ is a surface of revolution. Its tangent plane $x = 1$ intersects it in the

pair of orthogonal lines $(y+z)(y-z) = 0,\ x = 1$. Rotation around the z-axis gives two families of lines on the surface. Each tangent plane cuts the surface in two lines, one from each family.

Explicit parametrizations

Ellipsoid:
$x = aa \cdot \sin u \cos v,\ y = bb \cdot \sin u \sin v,\ z = cc \cdot \cos u,$

1-sheeted Hyperboloid:
$x = aa \cosh u \cos v,\ y = bb \cosh u \sin v,\ z = cc \sinh u,$

2-sheeted Hyperboloid (2nd sheet $z \to -z$):
$x = aa \sinh u \cos v,\ y = bb \sinh u \sin v,\ z = cc \cosh u,$

Elliptic Paraboloid:
$x = aa \cdot u \cos v,\ y = bb \cdot u \sin v,\ z = cc \cdot u^2,$

Hyperbolic Paraboloid:
$x = aa \cdot u,\ y = bb \cdot v,\ z = cc \cdot uv.$

H.K.

Ruled Surfaces *

Cylinders, Cones, 1-sheeted Hyperboloid, Hyperbolic Paraboloid, Helicoid, Right Conoid, Whitney Umbrella. *In other sections:* Double Helix, Möbius Strip.

Informally speaking, a **ruled surface** is one that is a union of straight lines (the rulings). To be more precise, it is a surface that can be represented parametrically in the form:

$$x(u, v) = \delta(u) + v * \lambda(u)$$

where δ is a regular space curve (i.e., δ' never vanishes) called the **directrix** and λ is a smooth curve that does not pass through the origin. Without loss of generality, we can assume that $|\lambda(t)| = 1$. For each fixed u we get a line $v \mapsto \delta(u) + v * \lambda(u)$ lying in the surface, and these are the rulings. (Some surfaces can be parameterized in the above form in two essentially different ways, and such surfaces are called **doubly-ruled surfaces**.)

A ruled surface is called a **cylinder** if the directrix lies in a plane P and $\lambda(u)$ is a constant direction not parallel to P, and it is called a **cone** if all the rulings pass through a fixed point V (the vertex).

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Two more interesting examples are quadratic surfaces:

The **Hyperboloid of One Sheet:**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

which is in fact doubly-ruled, since it can be given parametrically by:

$$x^+(u, v) = a(\cos(u) - v\sin(u)), b(\sin(u) + v\cos(u), cv)$$

and

$$x^-(u, v) = a(\cos(u) + v\sin(u)), b(\sin(u) - v\cos(u), -cv),$$

and the **Hyperbolic Paraboloid:**

$$(x, y, z) = (a\,u, b\,v, c\,u\,v) = a(u, 0, 0) + v(0, b, c\,u).$$

Another interesting ruled surface is a *minimal surface*: the **Helicoid**, $aa = 0$, (Catenoid, $aa = \pi/2$) in the family:

$$\begin{aligned} F(u,v) = &\ bb\sin(aa)\big(\cosh(v)\cos(u),\ \cosh(v)\sin(u),\ v\big) \\ &+ bb\cos(aa)\big(\sinh(v)\sin(u),\ -\sinh(v)\cos(u),\ u\big) \\ = &\ \sin(aa)\big((0,0,bb\,v) + bb\cosh(v)(\cos(u),\sin(u),0)\big) \\ &+ \cos(aa)\big((0,0,bb\,u) + bb\sinh(v)(\sin(u),-\cos(u),0)\big). \end{aligned}$$

A ruled surface is called a (generalized) **right conoid** if its rulings are parallel to some plane, P, and all pass through a line L that is orthogonal to P. *The* **Right Conoid** is given by taking P to be the xy-plane and L the z-axis:

Parametrized: $F(u,v) = (v\cos u, v\sin u, 2\sin u)$,

Implicitly: $(\frac{x}{y})^2 - \frac{4}{z^2} = 1.$

This surface has at $(\sin(u) = \pm 1, v = 0)$ two pinch point singularities. The `default morph` in 3DXM

deforms the **Right Conoid** to a **Helicoid**

so that the two stable pinch point singularities disappear, at the final moment, through two unstable singularities:

$$F_{aa}(u,v) = (\, v\cos(u), v\sin(u), 2aa\sin(u) + (1-aa)u\,)\,.$$

Famous for such a singularity is the **Whitney Umbrella**, another right conoid with rulings parallel to the x-y-plane:

$$F(u,v) = (u\cdot v, u, v\cdot v), \;\; \textit{implicitly: } x^2 - y^2 z = 0.$$

Again the `default morph` emphasizes the visualization of the singularity by embedding the **Whitney Umbrella** into a family of ruled surfaces, which develop a second pinch point singularity that closes the surface at the top:

$$F_{aa}(u,v) = \begin{pmatrix} u\cdot(aa\cdot v + (1-aa)\sin(\pi v)) \\ u \\ aa\cdot v^2 - (1-aa)\cos(\pi v) \end{pmatrix}.$$

R.S.P.

Monkey Saddle, Torus, Dupin Cyclide *

The **Monkey Saddle** is a saddle shaped surface with *three* down valleys, allowing the two legs and the tail of the monkey to hang down. At its symmetry point both principal curvatures are 0, and, this umbilic point is the simplest singularity of a curvature line field. Choose in the Action Menu: `Add Principal Curvature Fields`; in `Wireframe Display` the parameter lines are omitted, the curvature line fields (or one of them) represent the surface.
Its *Parametrization* as graph of a function is

$$F_{Monkey}(u,v) = (aa \cdot v,\ bb \cdot u,\ cc \cdot (u^3 - 3uv^2)).$$

In Geometry the word **Torus** usually implies a surface of revolution; often a circle in the x-z-plane ist rotated around the z-axis. In 3DXM an ellipse with axes bb, cc is rotated, its midpoint rotates in the x-y-plane on a circle of radius aa. The following *Parametrization* is used:

$$F_{Torus}(u,v) = \begin{pmatrix} (aa + bb \cdot \cos u) \cos v \\ (aa + bb \cdot \cos u) \sin v \\ cc \cdot \sin u \end{pmatrix}$$

Note that the parameter lines are principal curvature lines, see Action Menu: `Add Principal Curvature Fields`.

The **Torus** is also visualized among the Implicit Surfaces, we derive its equation. In the x-z-plane we have two ellipses

* This file is from the 3D-XplorMath project. Please see:

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and we multiply their equations:

$$
\begin{aligned}
&((\frac{x-aa}{bb})^2+(\frac{z}{cc})^2-1)\cdot((\frac{x+aa}{bb})^2+(\frac{z}{cc})^2-1)\\
=&(\frac{x^2-aa^2}{bb^2})^2+2(\frac{x^2+aa^2}{bb^2})((\frac{z}{cc})^2-1)+((\frac{z}{cc})^2-1)^2.
\end{aligned}
$$

For the rotation around the z-axis we have to replace x by $r=\sqrt{x^2+y^2}$. The second expression avoids square roots.

Implicit Equation of the **Torus**:

$$
f_{Torus}(\vec{x})=f(r,z)=0 \text{ with } r=\sqrt{x^2+y^2} \quad \text{and}
$$

$$
f(r,z):=
$$

$$
(\frac{r^2-aa^2}{bb^2})^2+2(\frac{r^2+aa^2}{bb^2})((\frac{z}{cc})^2-1)+((\frac{z}{cc})^2-1)^2.
$$

The **Cyclides of Dupin** are obtained by inverting the above torus in a sphere. The sphere of inversion has its center $\vec{m}=(dd,0,ee)$ in the x-z-plane and has radius ff. The `Default Morph` moves the center closer to the torus. Note that inversions map *curvature lines to curvature lines*.

The Inversion: $\vec{x}\mapsto \mathrm{Inv}(\vec{x}):=\dfrac{ff^2(\vec{x}-\vec{m})}{|\vec{x}-\vec{m}|^2}+\vec{m}+\begin{pmatrix}0\\0\\hh\end{pmatrix}$,

Parametrization: $F_{Cyclide}(\vec{x}):=\mathrm{Inv}(F_{Torus}(\vec{x}))$,

Implicit Equation: $f_{Cyclide}(\vec{x}):=f_{Torus}(\mathrm{Inv}^{-1}(\vec{x}))=0.$

Lissajous, Double Helix, Column, Norm 1 Family *

The French mathematician Jules Antoine Lissajous (1882-1880) studied vibrating objects by reflecting a spot of light of them, so that the various modes of vibration gave rise to *Lissajous curves*, see `Plane Curve Category`. Lissajous Space Curves and **Lissajous Surfaces** are a natural mathematical generalization. We use the *Parametrization:*

$$F_{Lissajous}(u,v) = \begin{pmatrix} \sin u \\ \sin v \\ \sin((dd - aa\,u - bb\,v)/cc) \end{pmatrix}.$$

The `Default Morph` joins a surface with tetrahedral symmetry and conical singularities and a surface with cubical symmetry and 12 pinch point singularities.

The **Double Helix** is a reminder of the famous double helix from genetics. For playing purposes there are two more parameters with default values $dd = 0, ee = 0$. We use the *parametrization*:

$$AA := aa + dd\,u,\ \ \alpha := (1 - ee\,u)u,$$

$$F_{DblHelix}(u,v) = \begin{pmatrix} AA((1-v)\cos\alpha + v\cos(\alpha + bb\,\pi)) \\ AA((1-v)\sin\alpha + v\sin(\alpha + bb\,\pi)) \\ cc\,u - 3.5 \end{pmatrix}.$$

This is a family of *ruled surfaces*: try the `Default Morph,` it varies the limits of the parameter v. With $bb = 1$ we get

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the Helicoid. The default parameters have been taken from Watson-Crick and give a reasonably good representation of molecular DNA. Dick Palais' biologist friend Chandler Fulton suggested the example and helped to get it right, many thanks.
We suggest to select `Move Principal Curvature Circles` from the Action Menu; this is seen best in `Point Cloud Display` from the View Menu.

Column Surface is used here in the sense of an architectural column, see the article by Marty Golubitzky and Ian Melbourne at

http://www.mi.sanu.ac.rs/vismath/golub/index.html

The order of the rotational symmetry around the z-axis is chosen with the 3DXM-parameter ii. Additional symmetry types can be selected with the parameter $hh = 1, \ldots, 6$; for other values of hh the unsymmetrized column shape is given by a formula that depends on the parameters $aa, \ldots, gg, ii$ and on the coordinates (θ, z). The formula is not determined by geometric properties, but is intended for playing.
The `Default Morph`, with $hh = 0$, varies the shape only mildly.

The **Norm 1 Family** is defined by the *implicit equation:*

$$f(x, y, z) = (|x|^p + |y|^p + |z|^p)^{1/p} = 1, \qquad 0 < p < \infty.$$

We get at $p = 1$ an *Octahedron*, at $p = 2$ a *Sphere* and at $p = \infty$ a *Cube*. For $1 \leq p \leq \infty$ these surfaces can be

viewed as the unit sphere in $\mathbb{R}^3$ for a Banach metric determined by p.
We *parametrize* these surfaces by spherical polar coordinates:

$$xp := \sin v \cos u,\ \ yp := \sin v \sin u,\ \ zp := \cos v,$$
$$x := \operatorname{sign}(xp)|xp|^e,\ \ y := \operatorname{sign}(yp)|yp|^e,\ \ z := \operatorname{sign}(zp)|zp|^e.$$

To obtain a reasonable family we set the exponent e in terms of the `default morphing` parameter ee as follows:

$$e := 1 + \tan(ee), \quad -\pi/4 < ee < \pi/2.$$

We obtain the Sphere at $ee = 0$,
the Octahedron at $ee = \pi/4$.
Numerical reasons prevent computation at $ee = -\pi/4$ (anyway a degenerate surface) and at $ee = \pi/2$, the Cube.
Already where we stop the computation the Pascal-values of sin near π had to be improved.

Snail Shell Surface *

These snail-like surfaces are included for their entertaining shapes. Try making one of your own. In spite of their complicated appearance, the snail surfaces are constructed as *one-parameter families of circles* $u \mapsto C_v(u)$. First we introduce two auxiliary variables. The surface parameter v is changed by a quadratic term that permits closing the snails at the top. The parameter ee controls the size of the opening of the snail (default $ee = -2$):

$$vv := v + (v + ee)^2/16.$$

The second variable controls the radius of the circles:

$$s := \exp(-cc \cdot vv). \quad \text{(Note that } s \text{ is a function of } v.\text{)}$$

The circles $u \mapsto C_v(u)$ of radius $s \cdot bb$ lie in an r-y-plane:

$$r := \quad s \cdot aa \ + \ s \cdot bb \cdot \cos(u),$$

$$y := dd(1 - s) + s \cdot bb \cdot \sin(u).$$

The parameter dd controls the length of the snail from top to bottom. And the other two coordinates in $\mathbb{R}^3$ are

$$x := r \cos(vv),$$

$$z := r \sin(vv),$$

so that the plane of the circle C_v also rotates with v.
Advice: Make only **small** changes to cc and keep $bb \geq aa$. The `Default Morph` varies dd and adjusts bb a little.

T.K.

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

Dirac Belt and Feynman Plate Tricks *

Dirac invented the famous belt trick to demonstrate a property of the motions of Euclidean Space that is indeed difficult to imagine without visual help.

The trick is performed with a strip, or belt, that is initially parallel to the screen. The orthogonal projection of the performance looks as follows: The left end of the belt stays fixed, the right end moves around the left end in a circular motion. It is important that the moving end **stays parallel** to the fixed end through the whole trick (parallel means: the final edge and the final normal each stay parallel to the initial edge and initial normal). One observes with surprise:
After moving the right end once around the circle the belt is twisted twice. After the second circular move the belt is untwisted (as it was initially).

The trick is shown in stereo because it is impossible that the belt stays in its initial plane when the ends are moved as described. It is important to visualize how the different parts of the belt move **vertically** to the screen.

It will be no surprise to observe that the first circular movement – when looked at in 3D – is different from the second circular movement. During the first circular movement the

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middle part of the belt moves vertically to be **in front** of the screen while the two ends stay on the screen. During the second circular movement it is the other way round: the middle part of the belt moves vertically to be **behind** the screen while the ends continue to be at their fixed vertical position. As soon as one can fix this image in one's mind it is obvious how the circular motion with parallel ends produces the twist of the belt.

Another instance of the same property of the Group of Euclidean motions is the **Feynman Plate Trick** or *Waiter's Cup Trick* : It is possible to continuously rotate a cup on ones horizontal hand in the **same** direction if during the first rotation the hand is above elbow height, during the second rotation below elbow height and so on, alternatingly above and below elbow height. Namely, imagine that the shoulder is the fixed end of the belt and the always horizontal – but continuously rotating – middle part of the belt is the hand. Choose `Do Plate Trick` from the Animation Menu to see half the belt performing the trick.

Some people cannot believe what they see. In such a case one can switch to `Monocular Vision` and `Orthographic Projection` in the View Menu to watch the belt – without at all observing the motion vertical to the screen. Or try `Patch Display` for a solid belt.

B.P.

Clifford Tori *

a) Parametrized by Curvature Lines, b) Hopf-fibered

Clifford Tori are embeddings of the torus into the unit sphere $\mathbb{S}^3$ of $\mathbb{R}^4$, by $(u,v) \to F(u,v) := (w,x,y,z)$, where

$$F_{\text{Clifford}}(u,v) = \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}(u,v) = \begin{pmatrix} \cos\alpha\cos u \\ \cos\alpha\sin u \\ \sin\alpha\cos v \\ \sin\alpha\sin v \end{pmatrix}$$

$\alpha :=\ aa + bb\sin(ee \cdot 2\,v),\ bb \neq 0$ for Bianchi-Pinkall Tori.

(Note that this is, for $bb = 0$, the product of a circle in the (w,x) plane with a second circle in the (y,z) plane, and so is clearly flat.) To get something that we can see in $\mathbb{R}^3$, we stereographically project $\mathbb{S}^3 \mapsto \mathbb{R}^3$; i.e., the Clifford tori in $\mathbb{R}^3$ are the embeddings $(u,v) \to P(F(u,v))$, where $P\!:\!\mathbb{S}^3 \to \mathbb{R}^3$ is stereographic projection.

We take as the center of the stereographic projection map the point $(\cos(cc\,\pi), 0, \sin(cc\,\pi), 0)$. Varying cc deforms a torus of revolution through cyclides. The `Default Morph` varies aa, hence changes the ratio of the two circles.

Another morph, `Conformal Inside-Out Morph` (also in the Animation Menu), is in $\mathbb{R}^4$ a rotation (parametrized by $0 \le ff \le 2\pi$), that moves the torus through the center of the stereographic projection. The image in $\mathbb{R}^3$ therefore passes through infinity: we see a torus with one puncture that has a flat end. It looks like a plane with a handle.

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The Clifford tori (in $\mathbb{S}^3$) are fibered by *Great Circles*, the Hopf fibers, $u + v = const$. These Great Circles are of course the *asymptote lines* on the tori. We show two versions of the stereographically projected Clifford tori: a) parameterized by curvature lines and b) by Hopf fibers. (To get the explicit parametrization of the latter, take $F(u+v, u-v)$ in the above formulae.)
The classical Clifford Torus corresponds to $\alpha = aa = \pi/4$. It has maximal area among the family and divides $\mathbb{S}^3$ into two congruent solid tori. But the other torus-leaves of the foliation, obtained by varying aa, are also interesting. All of them are foliated by *Clifford-parallel* great circles and hence flat. They are special cases of the flat Bianchi-Pinkall Tori in $\mathbb{S}^3$ (visible after stereographic projection) and discussed in more detail in their ATO ("About This Object..."), see the Documentation Menu.
Why is the ff-morph a `Conformal Inside-Out Morph`? A compact surface divides $\mathbb{R}^3$ in two components and the bounded component is called the inside. One surface of the family, the once punctured torus that passes through infinity, divides $\mathbb{R}^3$ into two congruent unbounded components. This surface has no inside and at this moment in the deformation inside and outside are interchanged. The 180 degree rotation in the rotation family is, on the $\pi/4$-torus, a conformal anti-involution. It has a Hopf fiber as connected fixed point set. Use in the Action Menu `Surface Coloration` and choose the default `two-sided user coloration` which emphasizes the fixed fiber.

Hopf Fibration and Clifford Translation* of the 3-sphere

See Clifford Tori and their discussion first.

Most rotations of the 3-dimensional sphere $\mathbb{S}^3$ are quite different from what we might expect from familiarity with 2-sphere rotations. To begin with, most of them have no fixed points, and in fact, certain 1-parameter subgroups of rotations of $\mathbb{S}^3$ resemble *translations* so much, that they are referred to as *Clifford translations*. The description by formulas looks nicer in complex notation. For this we identify $\mathbb{R}^2$ with $\mathbb{C}$, as usual, and multiplication by $\mathbf{i}$ in $\mathbb{C}$ represented in $\mathbb{R}^2$ by matrix multiplication by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Then the unit sphere $\mathbb{S}^3$ in $\mathbb{R}^4$ is given by:

$$\begin{aligned}\mathbb{S}^3 &:= \{p = (z_1, z_2) \in \mathbb{C}^2;\ |z_1|^2 + |z_2|^2 = 1\} \\ &\sim \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4;\ \sum (x_k)^2 = 1\}.\end{aligned}$$

And for $\varphi \in \mathbb{R}$ we define the Clifford Translation $C_\varphi : \mathbb{S}^3 \to \mathbb{S}^3$ by $C_\varphi(z_1, z_2) := (e^{\mathbf{i}\varphi} z_1,\ e^{\mathbf{i}\varphi} z_2)$.

The orbits of the one-parameter group C_φ are all great circles, and they are equidistant from each other in analogy to a family of parallel lines; it is because of this behaviour that the C_φ are called Clifford translations.

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But in another respect the behaviour of the C_φ is quite different from a translation – so different that it is difficult to imagine in $\mathbb{R}^3$. At each point $p \in \mathbb{S}^3$ we have one 2-dimensional subspace of the tangent space of $\mathbb{S}^3$ which is orthogonal to the great circle orbit through p. A Euclidean translation would simply translate these normal spaces into each other, but a Clifford translation rotates them so that the velocity of the translation along the orbit is equal to the angular velocity of the rotation of the normal spaces. This normal rotation is responsible for a very curious fact which is illustrated by the image in 3DXM:

Any two orbits are linked !

The fact that any two orbits are equidistant permits us to make the set of orbits into a metric space., and one can check that this space is isometric to the sphere of radius one-half in $\mathbb{R}^3$. Therefore one can map $h : \mathbb{S}^3 \to \mathbb{S}^2$ by mapping $p \in \mathbb{S}^3$ to its orbit, identified as a point of $\mathbb{S}^2$, and one can write this mapping in coordinates as:

$$h(z_1, z_2) = (|z_1|^2 - |z_2|^2, \mathrm{Re}\ (z_1 z_2), \mathrm{Im}\ (z_1 z_2))$$

This map h is called the *Hopf map* (or Hopf Fibration), and the orbits, the fibres of this map, are called *Hopf fibres.* It is named for Heinz Hopf, who studied it in detail, and found the completely unexpected fact that this map could not be deformed to a constant map.

The visualization in 3D-XplorMath shows *four tori each of which is made up of Hopf fibres.* We emphasize this with the coloration: each fibre has a constant colour and the colour varies with the distance of the fibres. One can see that any two of the four tori are linked, and one can also see that any two fibres on any one such torus are linked. Since 3D-XplorMath visualizes objects in $\mathbb{R}^3$, not in $\mathbb{S}^3$, before rendering the tori, we first map them into $\mathbb{R}^3$ using the following stereographic projection $\mathbb{S}^3 \to \mathbb{R}^3$

Stereographic Projection:

$$(x_1, x_2, x_3, x_4) \to (x_1, x_2, x_3)/(1 + x_4).$$

`The use of morphing parameters:`

The three cyclide-tori are made from the torus of revolution by *quaternion multiplication* in $\mathbb{S}^3$ with
$Qt2 = (\cos ee,\ 0,\ \sin ee \sin bb,\ \sin ee \cos bb)$,
$Qt3 = (\cos ff \cos cc,\ \sin ff,\ 0,\ \cos ff \sin cc)$,
$Qt4 = (\cos gg,\ 0,\ \sin gg \cos dd,\ \sin gg \sin dd)$.
One can therefore morph the first cyclide with the parameters bb, ee, the second with cc, ff and the third with dd, gg. The `Default Morph` changes the first two cyclides slightly while the third one performs a full rotation in $\mathbb{S}^3$ which moves it in $\mathbb{R}^3$ from its initial position through the torus of revolution and continues back to the initial position.

H.K.

Bianchi-Pinkall Flat Tori in $\mathbb{S}^3$ *

See Clifford Tori, Linked Tori and their discussions first.

1. Parameter Dependent Formulas in 3DXM

We can parametrize $\mathbb{S}^3$, considered as a submanifold of $\mathbb{C}^2$, by:

$$F(u, \alpha, v) = (\cos(\alpha)e^{iu}e^{iv}, \sin(\alpha)e^{iu}e^{-iv}),$$

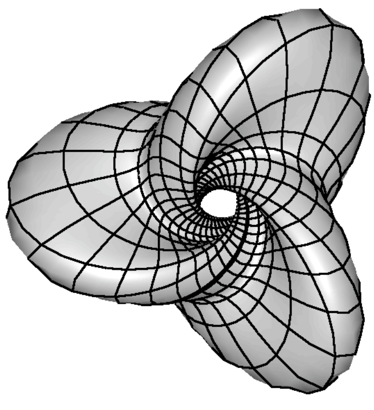
where $u \in [0, 2\pi)$, $\alpha \in [0, \pi/2]$, and $v \in [0, \pi]$. We will get the **Bianchi-Pinkall Tori** first as *flat tori* in $\mathbb{S}^3$ by taking α to be a function of v,

$$\alpha := aa + bb\sin(ee \cdot 2v)$$

(although the theory allows more general choices.) Next we *stereographically project* $\mathbb{S}^3$ from

$$p = (\cos(cc \cdot \pi), 0, \sin(cc \cdot \pi), 0)$$

to get **conformal** images of the flat tori in $\mathbb{S}^3$. The lines $v = const$ are circles, the stereographic images of the Hopf circles $u \mapsto F(u, \alpha, v)$.



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The `Default Morph` chooses $ee = 5$ and changes the amplitude bb thus increasing the five 'ears' of the torus. The `Range Morph` starts with a narrow band between two Hopf circles and widenes this band to the complete surface. Finally, the `Conformal Inside-Out Morph`, $0 \leq ff \leq 2\pi$, isometrically rotates $\mathbb{S}^3$ so that the Hopf circle $v = 0$ is the rotation axis. The stereographic image of this rotation is a conformal transformation of $\mathbb{R}^3 \cup \{\infty\}$ which "rotates" $\mathbb{R}^3$ around a circle on the pictured torus. In the case $aa = \pi/4$ we obtain for $ff = 0$ and $ff = \pi$ the same torus, but inside and outside interchanged. This is best viewed with the default `Two Sided User Coloration`, selectable from the `Surface Coloration` Submenu of the Action Menu.

2. Background and Explanations

The tori that we usually see are, from the point of view of complex analysis, rectangular tori, meaning that they have an orientation reversing symmetry and the set of fixed points of this symmetry has two components. (The better known tori of revolution have isometric reflections with **two** circles as fixed point sets.) Of course one tries to deform these tori to obtain non-rectangular ones. Obviously one can destroy the mirror symmetry, but this does not imply that one gets tori with a non-rectangular complex structure. The first proof, by Garcia, that one can embed all tori in $\mathbb{R}^3$ was non-constructive and difficult.

A simpler and constructive way to get tori with arbitrary conformal type was found by Pinkall, whose idea was to construct tori that are flat in $\mathbb{S}^3$ (and hence have an easy way to compute their conformal type from their flat geometry), and then stereographically project them to $\mathbb{R}^3$. While the resulting tori are no longer flat, this does preserve their conformal type.

The construction of flat surfaces in $\mathbb{S}^3$ goes back to 1894, when Bianchi classified all flat immersions in $\mathbb{S}^3$. In particular, he realized that the two families of asymptotic lines of a flat surface in $\mathbb{S}^3$ are left translations of a pair of curves that are either great circles or have constant torsion $+1$ and -1, respectively. The left translations arise by viewing $\mathbb{S}^3$ as the group of unit quaternions. An open problem for Bianchi was to determine when his flat surfaces were closed.

The first case when one of the curves is a great circle is of special interest for this problem. To explain why, we will need the Hopf fibration. Thinking of $\mathbb{S}^3$ as being part of $\mathbb{C}^2$, we can multiply points of $\mathbb{S}^3$ by e^{iu}, thus fibering $\mathbb{S}^3$ with circles, the Hopf circles, and the set of all such circles forms a metric space with distance being the distance between the Hopf circles in $\mathbb{S}^3$. As such it is isometric to a 2-dimensional sphere of radius $1/2$. We thus obtain a natural projection $\mathbb{S}^3 \to \mathbb{S}^2$, the Hopf map. It can be written as $(z_1, z_2) \mapsto z_1/z_2$, where we interpret the range as the Riemann sphere $\hat{\mathbb{C}}$. Moreover, Hopf circles are mapped to Hopf circles by left translations.

Now suppose we have a flat surface in $\mathbb{S}^3$ where one of the generating curves is a great circle. We can arrange $\mathbb{S}^3$ so that this great circle is part of the Hopf fibration, and thus all curves of the same family of asymptotic lines are Hopf circles. The surface in $\mathbb{S}^3$ is thus invariant under the Hopf action and projects to a curve in $\mathbb{S}^2$ under the Hopf map. Vice versa, the preimage of a curve in $\mathbb{S}^2$ under the Hopf map yields a flat surface in $\mathbb{S}^3$. In case the curve in $\mathbb{S}^2$ is closed, the surface in $\mathbb{S}^3$ is a flat torus. (The explanation so far is described in more detail in Spivak IV, p. 139ff.)

Pinkall found a simple way to determine the conformal type of the flat torus in terms of the geometry of the curve in $\mathbb{S}^2$ — in particular it was then easy to see that **all** possible conformal types can occur.

3. Visualizing Parts of the Theoretic Description

We cannot visualize $\mathbb{S}^3$ in such a way that all distances are preserved. We will use stereographic projection from $p = (\cos(cc \cdot \pi), 0, \sin(cc \cdot \pi), 0)$ to map $\mathbb{S}^3 - \{p\}$ one-to-one onto $\mathbb{R}^3$. Recall that: angles are not changed by stereographic projection, circles are mapped to circles or straight lines, and the images of great circles meet the equator sphere in antipodal points, so many properties of $\mathbb{S}^3$ get represented in geometrically comprehensible ways.
Our parametrization F of $\mathbb{S}^3$ emphasizes the Hopf fibration since the great circles $u \mapsto F(u, \alpha, v)$ are indeed the orbits of the Hopf-action of $\mathbb{S}^1$ on $\mathbb{S}^3$, given by $(u, p) \mapsto e^{iu}p$. Each such "Hopf Fiber" lies in one of the parallel tori $\alpha =$

constant, and the great circles $\alpha \mapsto F(u, \alpha, v)$, meet these ($\alpha =$ constant)-tori orthogonally, so that α measures the distance between them.

We get all the Hopf circles on each α-torus for $0 \leq v \leq \pi$, except that those tori degenerate to just one Hopf circle if $\alpha = 0$ or $\alpha = \pi$. This makes it plausible that $(\alpha, 2v)$ are polar coordinates on the metric space of Hopf circles, on the image $\mathbb{S}^2$ of the Hopf map.

Pinkall has observed that the closed curves on this image sphere, in polar coordinates given as: $(\alpha(s), 2v(s))$, (with $\alpha(s)$ never equal to 0 or $\pi/2$) allow one to write down immersed tori in $\mathbb{S}^3$ as:

$$(u, s) \mapsto (F(u, \alpha(s), v(s)).$$

For example taking $\alpha(s) = \pi/4$ gives the "Clifford Torus" in $\mathbb{S}^3$, a minimal embedding of the square torus. For other constant $\alpha(s)$ in $(0, \pi/2)$ one gets the above parallel family of α-tori, the lengths of their two orthogonal generators are $2\pi \cos(\alpha)$ and $2\pi \sin(\alpha)$.

On all of these tori we still have that the parameter lines $s =$ constant are Hopf-Fibers, and since these are equidistant (as orbits of an isometric action) it follows that the metric is flat. Pinkall proved that length and area of the curve in $\mathbb{S}^2$ determine the conformal structure of the torus in $\mathbb{S}^3$, hence in $\mathbb{R}^3$, and that all conformal structures occur.

Observe that the usual tori of revolution in $\mathbb{R}^3$ are all rectangular, and most of the Bianchi-Pinkall tori shown by 3D-XplorMath are very different from these. The tori with aa $= \pi/4$ are all rhombic, because they can be rotated into themselves by 180° rotations (in $\mathbb{S}^3$, not in $\mathbb{R}^3$) around any of the Hopf-Fibers on them. A cyclic morph with $0 \le ff \le 2\pi$ rotates around the circle $v = 0$ (we see of course the stereographic image of that rotation). For $ff = \pi$ we get an anti-involution of the torus with the circle as the (connected) fixed point set—only rhombic tori have such anti-involutions. (The square torus is rectangular and rhombic.) In the rhombic case $aa = \pi/4$ we get for $ff = \pi/2$ and $ff = 3\pi/2$ surfaces in $\mathbb{S}^3$ that pass through p so that the stereographic images in $\mathbb{R}^3$ pass through ∞ — otherwise we could not turn the torus inside out continuously.

The program takes $\alpha(v) := aa + bb \sin(\mathrm{ee}\, 2v)$ (with $ee = 3$ for the default image and $ee = 5$ for the default morph), allowing rather different examples.

Again, these tori are shown in $\mathbb{R}^3$ by using the (conformal) stereographic projection of $\mathbb{S}^3 \setminus \{p\} \to \mathbb{R}^3$, where $p = (\cos(cc \cdot \pi), 0, \sin(cc \cdot \pi), 0)$. Morphing cc therefore gives other images of $\mathbb{S}^3$, in particular other conformal images of these tori.

H.K.

Möbius Strip and **Klein Bottle** *

Other non-orientable surfaces in 3DXM:
Cross-Cap, Steiner Surface, Boy Surfaces

The **Möbius Strip** is the simplest of the non-orientable surfaces. On all others one can find Möbius Strips. In 3DXM we show a family with ff halftwists (non-orientable for odd ff, $ff = 1$ the standard strip). All of them are *ruled surfaces*, their lines rotate around a central circle. Möbius Strip *Parametrization:*

$$F_{Möbius}(u,v) = \begin{pmatrix} aa(\cos(v) + u\cos(ff \cdot v/2)\cos(v)) \\ aa(\sin(v) + u\cos(ff \cdot v/2)\sin(v)) \\ aa\, u \sin(ff \cdot v/2) \end{pmatrix}.$$

Try from the View Menu: `Distinguish Sides By Color`. You will see that the sides are not distinguished—because there is only one: follow the band around.

We construct a **Klein Bottle** by curving the rulings of the Möbius Strip into figure eight curves, see the Klein Bottle *Parametrization* below and its `Range Morph` in 3DXM.

$$w = ff \cdot v/2$$

$$F_{Klein}(u,v) = \begin{pmatrix} (aa + \cos w \sin u - \sin w \sin 2u)\cos v \\ (aa + \cos w \sin u - \sin w \sin 2u)\sin v \\ \sin w \sin u + \cos w \sin 2u \end{pmatrix}.$$

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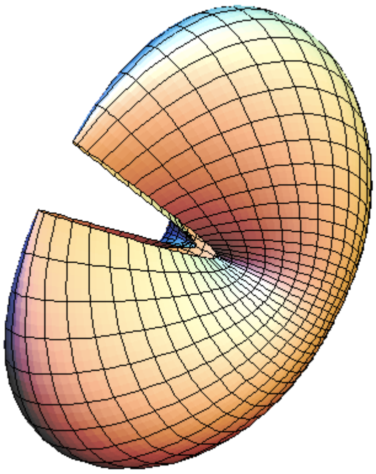
Actually, there are *three different Klein Bottles* which one cannot deform into each other through immersions. The best known one has a reflectional symmetry and looks like a veird bottle. The other two are mirror images of each other. Along the central circle one of them is left-rotating the other right-rotating. See the `Default Morph` of the *Möbius Strip* or of the *Klein Bottle*: both morphs connect a left-rotating to a right-rotating surface.

On the *Boy Surface* one can see different Möbius Strips. The `Default Morph` begins with an equator band which is a Möbius Strip with *three halftwists*. As the strip widens during the deformation it first passes through the triple intersection point and at the end closes the surfaces with a disk around the center of the polar coordinates.
Moreover, each meridian is the centerline of an *ordinary Möbius Strip*. Our second morph, the `Range Morph`, rotates a meridian band around the polar center and covers the surface with embedded Möbius Strips. - We suggest to also view these morphs using `Distinguish Sides By Color` from the View Menu.

On the Steiner Surface and the Cross-Cap the Möbius Strips have self-intersections and are therefore more difficult to see. The `Default Morph` for the *Steiner Surface* emphasizes this unusual Möbius Strip. - The `Range Morph` of the *Cross-Cap* shows a family of embedded disks, except at the last moment, when opposite points of the boundary are identified, covering the self-intersection segment twice.

The Cross-Cap and Steiner's Roman Surface *



Opposite boundary points of this embedded disk are identified along a segment of double points when the surface is closed to make the Cross-Cap. Pinchpoint singularities form at the endpoints of the selfintersection segment.

In the 19th century images of the projective plane where found by restricting quadratic maps $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ to the unit sphere. For example a **Cross-Cap** is obtained with $f(x, y, z) = (xz, yz, (z^2 - x^2)/2$, and **Steiner's Roman Surface** with $f(x, y, z) = (xy, yz, zx)$.
Parametrizations follow by restricting to a parametrized sphere $F_{Sphere}(u, v) = \big(\sin u \cos v,\ \sin u \sin v,\ \cos u\big)$.

Steiner's surface has three self-intersection segments and six pinchpoint singularities. The `Default Morph` emphasizes a (self-intersecting) Möbius Band on this surface.

Cross-caps occur naturally as a family by a differential geometric construction. Consider at a point p of positive curvature of some surface the family of all the normal curvature circles at p. They form a cross-cap and the two

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pinchpoint sigularities are the points opposite to p on the two principal curvature circles. The parameters aa, bb in 3DXM are the two principal curvature radii. The `Default Morph` varies bb from $bb = 0.4aa$ to $bb = aa$, a sphere. A `Range Morph` starts by taking half of each normal curvature circle and slowly extends them to full circles.

To derive a parametrization of this family of cross-caps let e_1, e_2 be a principal curvature frame at p, let κ_1, κ_2 be the principal curvatures and $r_1 := 1/\kappa_1, r_2 := 1/\kappa_2$ the principal curvature radii at p. The normal curvature in the direction $e(\varphi) := e_1 \cos\varphi + e_2 \sin\varphi$ is
$\kappa(\varphi) := \kappa_1 \cos^2\varphi + \kappa_2 \sin^2\varphi$ with $r(\varphi) := 1/\kappa(\varphi)$.
In 3DXM we parametrize the circles by $u \in [-\pi, \pi]$ and use $v = \varphi$. Denoting the surface normal by n we get the family of normal circles as

$$r(v) \cdot (-n + n \cdot \cos u + e(v) \cdot \sin u),\ u \in [0, \pi],\ v \in [0, 2\pi].$$

Finaly we take $\{e_1, e_2, n\}$ as the (x, y, z) coordinate frame and allow translation by cc along the z-axis to get our

Parametrization of the normal curvature Cross-cap

$$\begin{aligned} r(v) &= r_1 r_2/(r_2 \cos^2 v + r_1 \sin^2 v), \\ x &= r(v) \cos v \sin u, \\ y &= r(v) \sin v \sin u, \\ z - cc &= r(v)(-1 + \cos u) = 2r(v) \sin^2(u/2), \\ & u \in [0, \pi],\ v \in [0, 2\pi]. \end{aligned}$$

To also get an *implicit equation* we observe
$y/x = \tan v$ and $z^2/(x^2+y^2) = \tan^2(u/2)$.
This leads to

$$\begin{aligned}
x^2/(x^2+y^2) &= \cos^2 v, \\
y^2/(x^2+y^2) &= \sin^2 v, \\
z^2/(x^2+y^2+z^2) &= \sin^2 u/2, \\
(x^2+y^2)/(x^2+y^2+z^2) &= \cos^2 u/2.
\end{aligned}$$

The first two of these equations eliminate v from $r(v)$. The third one eliminates u from $z/r(v) = 2\sin^2(u/2)$ and gives (with $r_1 = aa,\ r_2 = bb$) an

Implicit equation of the normal curvature Cross-cap

$$(\frac{x^2}{aa} + \frac{y^2}{bb})(x^2+y^2+z^2) + 2z(x^2+y^2) = 0.$$

Finally, replacing z by $z - cc$ will translate the cross-cap, e.g. $cc = 2bb$ puts the pinchpoint of the smaller curvature circle to the origin of $\mathbb{R}^3$.

H.K.

Boy Surfaces,
following **Apery** and **Bryant-Kusner** *

See Möbius Strip and Cross-Cap first.

All the early images of the projective plane in $\mathbb{R}^3$ had singularities, the **Boy Surface** was the first *immersion.* Since the projective plane is *non-orientable*, no embedding into $\mathbb{R}^3$ exists and *self-intersection curves* have to occur on the image. In fact, the self-intersection curve of the Boy surface is also *not* embedded, the surface has a *triple point.* Boy discovered the surface while working for his PhD under Hilbert. Boy's construction was differential topology work, his surface has no special local geometry.

Apery found an **Algebraic Boy Surface**. Moreover, his surface is covered by a 1-parameter family of ellipses. This is his *Parametrization*:

$$F_{Apery}(u,v) = \begin{pmatrix} \frac{\cos^2(u)\cos(2v)+\sin(2u)\cos(v)/\sqrt{2}}{\sqrt{2}-\sin(2u)\sin(3v)} \\ \frac{\cos^2(u)\sin(2v)-\sin(2u)\sin(v)/\sqrt{2}}{\sqrt{2}-\sin(2u)\sin(3v)} \\ \frac{\sqrt{2}\cos^2(u)}{(\sqrt{2}-\sin(2u)\sin(3v))} \end{pmatrix}.$$

This parametrization is obtained by restricting the following *even* map from $\mathbb{R}^3$ to $\mathbb{R}^3$ to the unit sphere:

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$$
\begin{aligned}
denom &:= (\sqrt{2} - 6xz)(x^2 + y^2) + 8x^3 z \\
F_x &:= ((x^2 - y^2)z^2 + \sqrt{2}xz(x^2 + y^2))/denom \\
F_y &:= (2xyz^2 - \sqrt{2}yz(x^2 + y^2))/denom \\
F_z &:= z^2(x^2 + y^2)/denom
\end{aligned}
$$

The image of the unit sphere is also an *image of the projective plane* since $(F_x, F_y, F_z)(-p) = (F_x, F_y, F_z)(p)$.

The `Default Morph` starts with a band around the equator, which is a Möbius Strip with *three* halftwists. The complete surface is obtained by attaching a disk (centered at the polar center). 3DXM supplies a second morph, `Range Morph` in the Animation Menu. It starts with a band around a meridian, which is another Möbius Strip with *one* halftwist. This Möbius Strip is moved over all the meridians, covering the surface with embedded Möbius Strips.

Bryant-Kusner Boy Surfaces are obtained by an inversion from a minimal surface in $\mathbb{R}^3$. The minimal surface is an immersion of $\mathbb{S}^2 - \{6 \text{ points}\}$ such that antipodal points have the same image in $\mathbb{R}^3$, so that the minimal surface is an image of the projective plane minus three points. The six punctures are three antipodal pairs, and the minimal surface has so called *planar ends* at these punctures. This is the same as saying that the Weierstrass-integrand has no residues, hence can be explicitly integrated. In this context it is important that the inversion of a planar end has a puncture that can be *smoothly* closed by adding one point.

The closing of the three pairs of antipodal ends thus gives a triple point on the smoothly immersed surface which is obtained by inverting the minimal surface.

As `Default Morph` and as `Range Morph` we took the same deformations as in the algebraic case. The first emphasizes the equator Möbius Strip with *three* halftwists, the second covers the surface with embedded Möbius Strips that have meridians as center lines.

A *Parametrization* is obtained by first describing the minimal surface as an image of the Gaussian plane, then invert it in the unit sphere. Parameter lines come by taking polar coordinates in the unit disk.

$MinSurf(z) := \mathrm{Re}\,(V(z)/a(z)) + (0,0,1/2)$, where
$a(z) := \big(z^3 - z^{-3} + \sqrt{5}\big)$ and
$V(z) := \big(i(z^2 + z^{-2}),\ z^2 + z^{-2},\ \frac{2i}{3}(z^3 + z^{-3})\big)$.
Finally the inversion:

$$Boy(z) := \frac{MinSurf(z)}{||MinSurf(z)||^2}.$$

H.K.