Ordinary Differential Equations
in 3D-XplorMath, a Visualization Program

One Dimensional, First Order:

1.) Logistic Equation
2.) Equation of Mass Action
3.) User 1D First Order ODE

One Dimensional, Second Order:

4.) Harmonic Oscillator
5.) Pendulum
6.) Forced Oscillator
7.) Forced Duffing Oscillator
8.) Forced Van Der Pol Oscillator
9.) User 1D Second Order ODE

Two Dimensional, First Order:

10.) Harmonic Oscillator
11.) Pendulum
12.) Linear
13.) Volterra-Lotka
14.) User 2D First Order ODE

Two Dimensional, Second Order:

15.) Coupled Oscillators
16.) Forced Oscillators
17.) Foucault Pendulum
18.) User 2D Second Order ODE

Three Dimensional, First Order:
19.) Linear
20.) Lorenz
21.) Rikitake 2-Disk Dynamo
22.) Rössler
23.) User 3D First Order ODE

Three Dimensional, Second Order:
24.) Coupled Oscillators
25.) Forced Oscillators
Charged Particles:
26.) Constant Magnetic Field
27.) Current in a Straight Wire
28.) Toroidal Magnetic Field
29.) Magnetic Dipole
30.) User Magnetic Field
31.) User 3D Second Order ODE
Central Force:
32.) Coulomb
33.) Power Law
34.) Yukawa
35.) Hooke's Law
36.) Higgs
37.) User Central Force
Lattice Models:
38.) Fermi-Pasta-Ulam
39.) Toda
40.) Discrete Sine-Gordon
41.) Tacoma Narrows Bridge

Forced Duffing Oscillator.

1. **What is it?**

What we shall call the *Forced Duffing Oscillator Equation* is the second order ODE for a single variable x,

$$\frac{d^2x}{dt^2} = -hh\,x - ii\,x^3 - aa\,\frac{dx}{dt} + bb\cos(cc\,t) \qquad (1)$$

whose solutions we display via the equivalent (non-autonomous) first order system in two variables, x and y:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -hh\,x - ii\,x^3 - aa\,y + bb\cos(cc\,t) \qquad (2)$$

which in turn can be made into an autonomous first order system in three variables, T, x and y:

$$\frac{dT}{dt} = 1,\ \frac{dx}{dt} = y,\ \frac{dy}{dt} = -hh\,x - ii\,x^3 - aa\,y + bb\cos(cc\,T). \quad (3)$$

We discuss the interpretation and significance of the five parameters, aa, bb, cc, hh, ii below. Their default values are: $aa = 0.25, bb = 0.3, cc = 1.0, hh = -1.0,$ and $ii = 1.0.$

2. **Why is it interesting?**

Here are two of the considerations that make the oscillator equation (1) worth studying. First, with appropriate choices of parameter values it reduces to a variety of mathematically and physically interesting oscillator models; some classical such as the harmonic oscillator (with and without damping and forcing) and others that are more exotic, such as the classic Duffing oscillator introduced by Duffing in 1918. By putting these together in a parametric family, we can investigate how various features of these systems behave as we move around in the parameter space. Secondly—and more importantly—it was in in the study of the Duffing Oscillator that symptoms of the phenomena we now call "chaos" and "strange attractor" were first glimpsed (although their significance was only appreciated later). By the Poincaré-Bendixson Theorem, three is the smallest dimension in which an autonomous system can exhibit chaotic behavior, and the Duffing system is so simple that it lends itself very easily to the study and visualization of the phenomena related to chaos.

3. **The Newtonian Particle Interpretation.**

Note that (1) becomes Newton's equation of motion for a particle of unit mass moving on the x-axis if we define the "force", $F(x, \frac{dx}{dt}, t)$, acting on the particle to be the right-hand side of (1). Let's interpret the various terms of F from this point of view.

If hh is positive then the term $-hh\,x$ by itself gives Hooke's Law for a spring, that "stress is proportional to strain" and the parameter hh has the interpretation of Hooke's proportionality factor between the extension of the spring, x, and the restoring force. If also $ii = 0$ then we have a pure Hooke's Law force that gives the Harmonic Oscillator, $\frac{d^2x}{dt^2} = -hh\,x$. But a real spring only satisfies Hooke's Law approximately, and the term $-ii\,x^3$ represents the next term in the Taylor expansion of the restoring force under the reasonable assumption that this force is an odd function of the spring extension, x. (If ii is positive it is called a "hardening" spring and if negative a "softening" spring.) For the classic Duffing Oscillator, hh is negative and ii is positive and there is not a good interpretation of the force in terms of a spring. Rather, the sum of the two terms $-hh\,x - ii\,x^3$ should be interpreted as the force on a particle that is moving in a double-well potential as we will discuss in more detail below.

The term $-aa\,\frac{dx}{dt}$ represents a "friction" force of the sort that would be experienced by a particle like a bullet traveling through air or a bead sliding on a wire; that is, assuming that the "damping" or "friction" coefficient aa is positive, it describes a force acting on the particle in the direction opposite to the velocity and with a magnitude that is proportional to the magnitude of the velocity.

Under the sum of the above terms of the force law F, the particle will (in general) oscillate back and forth—which of course is why it is called an oscillator—however if $aa > 0$ these oscillations will gradually die down as the kinetic energy is absorbed by friction. The final term in the force law, $bb\cos(cc\,t)$ is a periodic forcing term that will act on and perturb the motion of this oscillating particle, and we note that it is solely a function of the time and is independent of both the position and velocity of the particle. We will discuss a possible physical interpretation of this term later. The parameter bb is clearly the amplitude of this forcing term, i.e., its maximum magnitude, and the parameter cc is the angular velocity of its phase in radians per unit time, so that the period of the forcing term is $\frac{2\pi}{cc}$ and its frequency is $\frac{cc}{2\pi}$. As we shall see, it is the energy that is fed into the system by this forcing term that is essential for the interesting chaos related effects to occur. In fact the most interesting behaviors of solutions of (1) are present when all the above terms are present in F, that is when the oscillator is both forced and damped, and in fact the way damping and forcing can balance each other is crucial to understanding the general behavior of solutions. However we will begin by analyzing the simpler situation when both the damping and forcing terms are missing.

4. **The Undamped, Unforced Case.**

We now assume that aa and bb are both zero, so the force $F(x) = -hh\,x - ii\,x^3$ is a function of x alone. Now in one-dimension, whenever this is the case the force is *conservative*, that is, it is minus the derivative of a "potential" function, $U(x)$. Indeed, if we define $U(x) := -\int_0^x F(\xi)\,d\xi$, then clearly $F(x) = -U'(x)$. If as above we write $y := \frac{dx}{dt}$, define the kinetic energy by $K(y) := \frac{1}{2}y^2$ and define the Hamiltonian or total energy function by $H(x, y) := K(y) + U(x)$, then $\frac{dH}{dt} = y\,\frac{dy}{dt} + U'(x)\,\frac{dx}{dt} = y(\frac{dy}{dt} + U'(x))$. So, if Newton's Equation is satisfied, $\frac{dy}{dt} = \frac{d^2x}{dt^2} = F(x) = -U'(x)$, so $\frac{dH}{dt} = 0$. This of course is the law of conservation of energy: the total energy function $H(x, y)$ is constant along any solution of Newton's Equations. In one-dimension this provides at least in principle a way to solve Newton's Equation for any initial conditions $x = x_0$ and $y = y_0$ at time $t = t_0$. Namely, the path or orbit of the solution is a curve in the x-y plane, and by conservation of energy this curve is given by the implicit equation $H(x,y) = H(x_0,y_0)$. And since $\left(\frac{dx}{dt}\right)^2 = y^2 = 2K(y) = 2(H(x_0,y_0) - U(x))$, we find:

$$\frac{dt}{dx} = \frac{1}{\sqrt{2(H(x_0,y_0) - U(x))}},$$

so we can find t as a function of x by a quadrature, and then invert this relation to find x as a function of t.

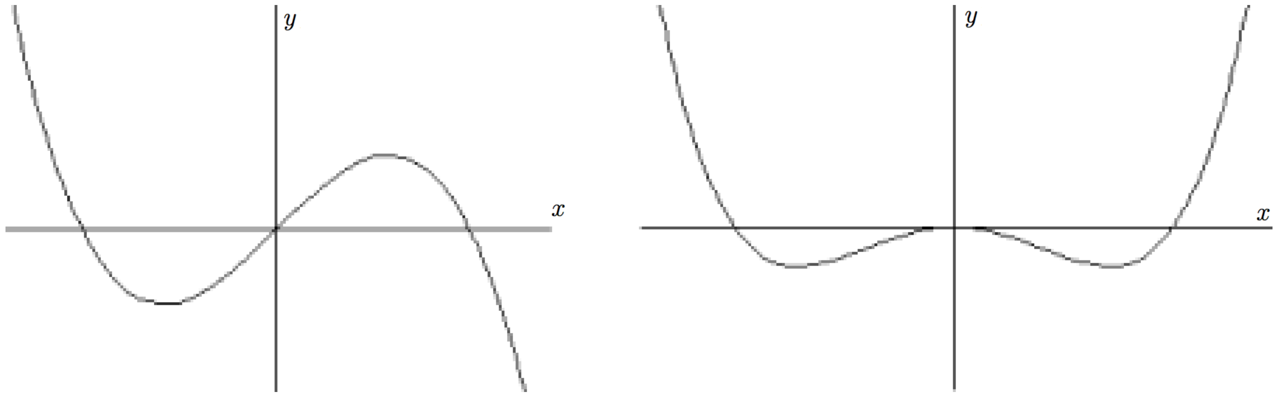
In the Harmonic Oscillator case, with $hh = 1$ and $ii = 0$, $U(x) = \frac{1}{2}x^2$ so $H(x,y) = \frac{1}{2}(x^2 + y^2)$, so the orbits are circles, and it is easy to carry out the above quadrature and inversion explicitly, to obtain $x(t) = x_0 \cos(t - t_0) + y_0 \sin(t - t_0)$.

5. The Universal "Sliding Bead on a Wire" Model.

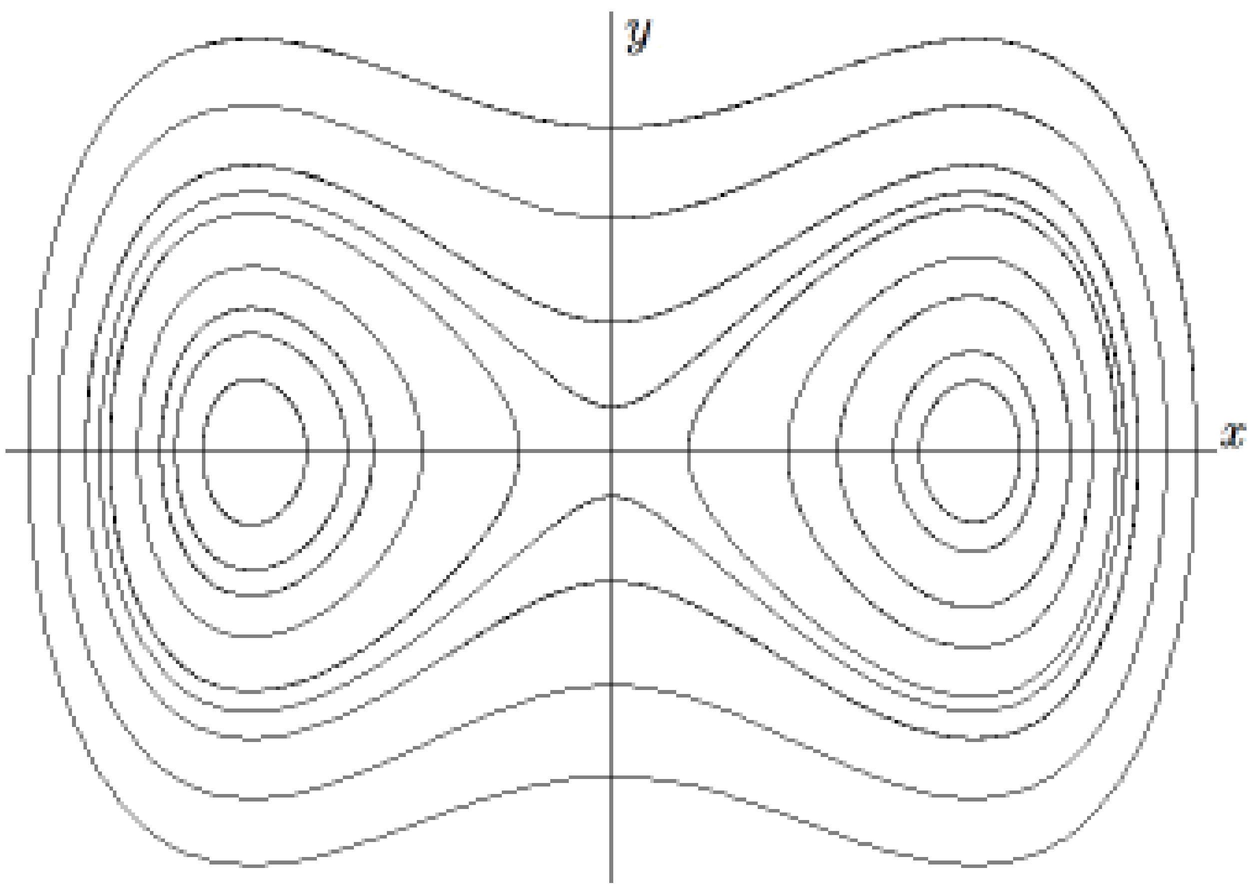
In one-dimension there is a highly intuitive physical model that makes it easy to visualize the motion of a particle under a given force. Moreover this model is "universal" in the sense that it works for all forces that are function of position only and hence, as we noted above, are of the form $F(x) = -U'(x)$ for some potential function U. Namely, imagine that we string a bead on a frictionless piece of wire that lies along the graph of the equation $y = U(x)$. If the bead has mass $m = 1$ and if we choose units so that g, the acceleration of gravity, equals one, then the gravitational potential of the bead is $mgh = h$ where h is its height. So if as usual we interpret the ordinate of a point as its height, then the gravitational potential of the bead when it is at the point $(x, y) = (x, U(x))$ is just $U(x)$, and the sliding motion of the bead along the wire under the attraction of gravity will exactly model whatever other system we started from!

In the case of the Harmonic Oscillator, where $F(x) = -x$ and $U(x) = \frac{1}{2}x^2$, the graph is the parabola, $y = \frac{1}{2}x^2$ and it is easy to imagine the bead oscillating back and forth along this parabola.

For the unforced and undamped Duffing Oscillator the force is $F(x) = -hh\,x - ii\,x^3$, where for simplicity in what follows we will assume that $ii > 0$ and $hh < 0$. The potential is $U(x) = \frac{hh}{2}x^2 + \frac{ii}{4}x^4$, which we note can be considered as the first two terms in the Taylor expansion for an arbitrary symmetric potential with local maximum at 0. It is easily checked that $\lim_{x\to\pm\infty} U(x) = +\infty$ and in addition to the local maximum at 0, there are two other critical points of U, at $x = \pm\sqrt{\frac{-hh}{ii}}$, where U has local minima. For the default values, $hh = -1$ and $ii = 1$, the force is $F(x) = x(1-x^2)$, and the potential is $U(x) = \frac{1}{4}x^2(x^2-2)$, so the local minima are at ± 1. We graph this force $F(x)$ and potential $U(x)$ below, and show a selection of the resulting orbits. It should be clear why U is called a double-well potential.



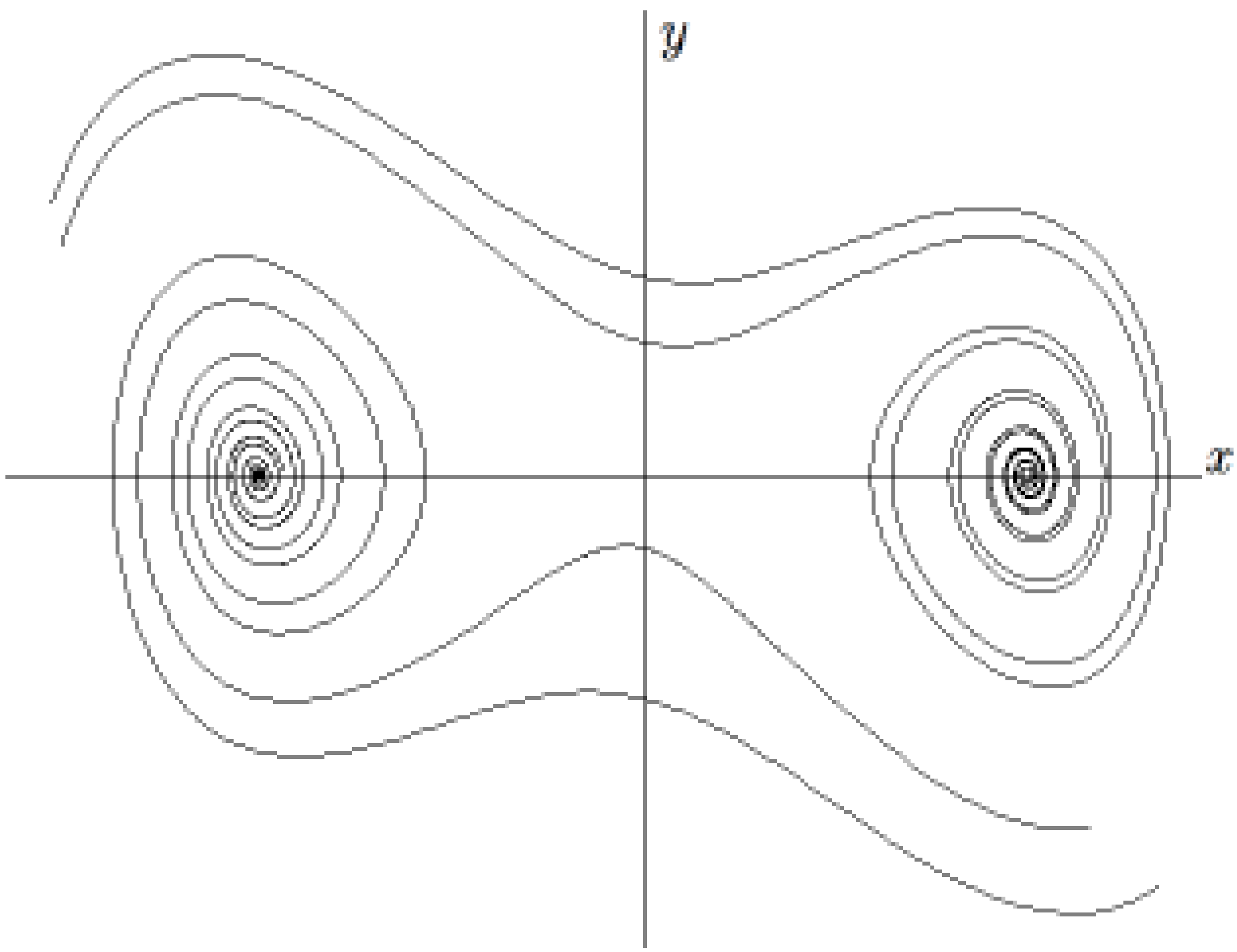
$F(x)$ $U(x)$



Some orbits of the Unforced, Undamped Duffing Oscillator

6. **The Unforced, Damped Duffing Oscillator.**

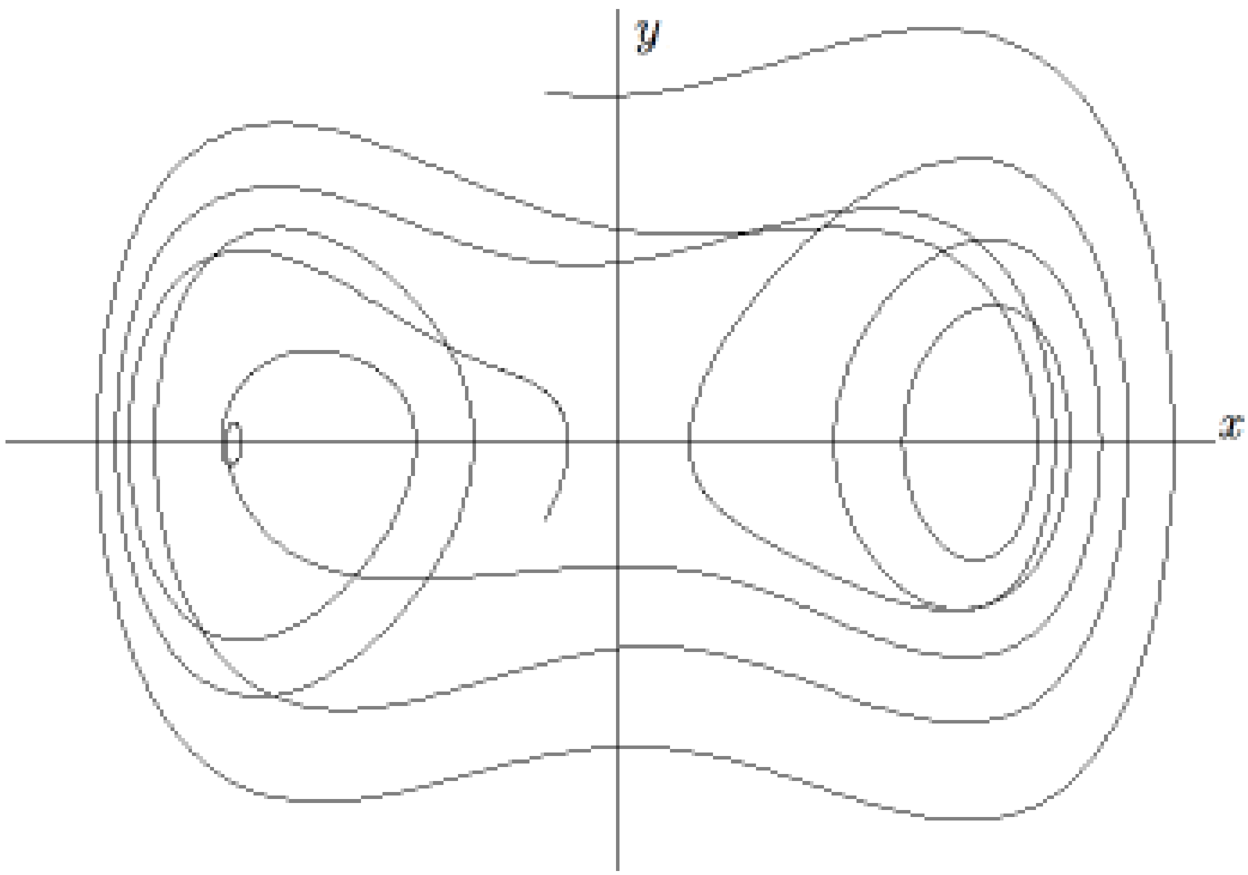
We now still assume $bb = 0$ (so there is no forcing) but we assume that $aa > 0$, so there is damping. In the bead on a wire picture, $aa\,\frac{dx}{dt} = aa\,y$ is the friction from the bead rubbing against the wire, and the force is now given by $F(x) = -U'(x) - aa\,y$. If we again calculate $\frac{dH}{dt}$ as we did above, we now find not $\frac{dH}{dt} = 0$ but instead $\frac{dH}{dt} = -aa\,y^2$. The result is that instead of the orbits of the bead in the x-y-plane being closed curves of constant total energy H, the energy decrease along the obits, and they cut across the $H =$ constant curves and spiral in towards the two minima of H at the bottom of the two potential wells. We show a selection of the resulting orbits below.



Some orbits of the Unforced, Damped Duffing Oscillator

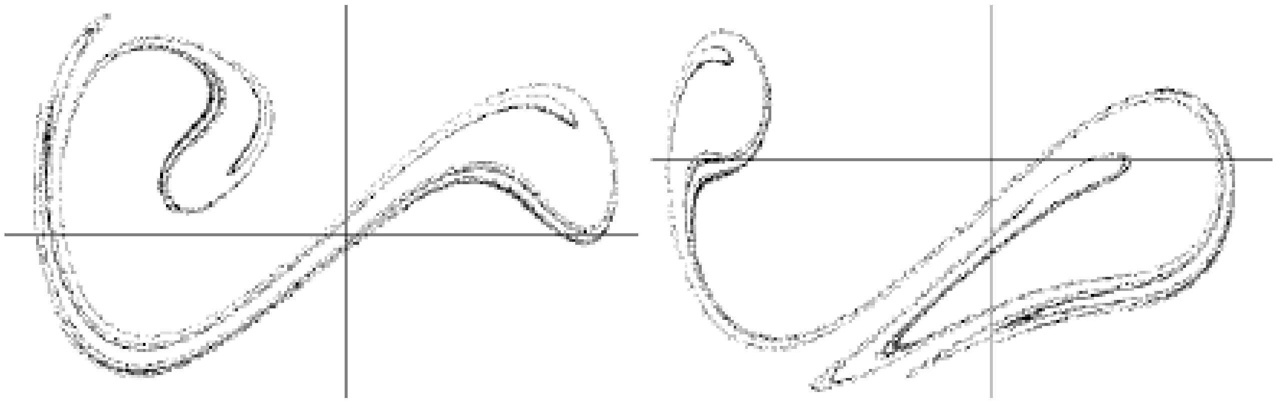
7. The Forced Duffing Oscillator.

We now add back the forcing term $bb\cos(cc\,t)$. First a word about how to interpret this force in the sliding bead picture. If we assume that there is an alternating electric field parallel to the x direction and with strength $\cos(cc\,t)$ at time t, then $bb\cos(cc\,t)$ will be the electric force felt by the bead if we give the bead an electric charge of magnitude bb.



Some orbits of the Forced, Damped Duffing Oscillator

8. Chaos, Strange Attractors, and Poincaré Sections.



Two Slices of the Duffing Attractor

About ODE 2nd Order: Charged Particles

The Motion of Charged Particles in Magnetic Fields

The path $p(t)$ of a particle with electric charge e and mass m in a magnetic field B is given by

$$m \cdot p''(t) = e \cdot p'(t) \times B(p).$$

(The right hand side is called the *Lorentz Force*.)

This implies that, for an arbitrary magnetic fields, B, the kinetic energy $E(t) = \frac{m}{2}\langle p',\ p'\rangle(t)$ is constant in time.

One should first convince oneself in the case of a

Constant Magnetic Field

that a particle can move tangentially to the field lines, in circles around the field lines and in helices around the field lines, i.e., in any linear combination of the first two special cases.

Put in *Settings, ODE Settings*:

$v_x = 0.003, v_y = 0.003, v_z = 0.5$, to obtain almost circles around the field lines.

And put $v_x = 0.2, v_y = 0.2, v_z = 0.001$ to obtain approximate straight lines parallel to the field lines. The *Default Settings* give a general helix. We will also try to understand charged particle motions in nonlinear fields by looking at such special cases.

We consider next motion of a charged particle in the:

Field of an Electric Current

along the x-axis. The field lines are circles parallel to the y-z-plane with centers on the x-axis. In this case, the *Default Settings* give initial conditions in the x-y-plane (a symmetry plane of the field) that are orthogonal to the field lines. The solution curves therefore remain in this plane, and are, for small velocities, *almost circles around the field lines*. But, because the absolute value of the field is decreasing with r, these solution curves are more strongly curved the nearer they are to the wire. They are therefore rolling curves with a translational period in the direction of the wire. (See *Plane Curves, Cycloid* and put $aa = 1$ and $bb = 6.5$ in the *Set Parameters* dialog.) If in *ODE Settings* one increases the velocity to $v_y = 0.5$, then the translational part is so large that the consecutive loops do not intersect.

We obtain solution curves which *almost follow the field lines* if we make the initial velocity tangential to the field lines and fairly small:

$v_x = 0, v_y = 0, v_z = 0.02,$

Time span = 450, Step-size = 0.2.

Now slowly increase v_z, e.g., to $v_z = 0.2$, to obtain another family of solutions *follows the field lines, but winding around them in small loops*.

Next in *Settings, ODE Settings*, put:

$v_x = 0.02, v_y = 0.02, v_z = 0.01$

leaving Time span = 450, Step-size = 0.2, as before.
Finally we increase the initial velocity to obtain solution curves that look fairly wild at first but can be seen to follow the pattern which we recognized for more special initial conditions, namely put
$v_x = 0.2, v_y = 0.2, v_z = 0.1$
to see solutions that follow helices with wide loops around them. *Try by all means to view this in stereo!*

Finally we consider the so-called *Störmer Problem*, namely the motion of charged particles in a Magnetic Dipole Field. Since the magnetic field of our Earth is a dipole field, such motions occur in the van Allan Belt when charged particles from the Sun's plasma meet the Earth. A dipole field $B(p)$ with a magnetic moment mm is given by:

$$B(p) = 3\langle mm,\ p\rangle \frac{p}{|p|^5} - \frac{mm}{|p|^3}.$$

The *Default ODE Settings* give a fairly general but somewhat complicated solution curve. To see solutions that *almost follow the field lines* use ODE Settings to set a small initial velocity tangential to the field lines, say $v_x = 0, v_y = 0, v_z = 0.05$. To see solutions that *almost circle the field lines in the equator plane of the dipole*, in the ODE Settings dialog, choose small initial conditions in the equator plane, e.g., $v_x = 0.1, v_y = 0.1, v_z = 0$. The resulting curves are close to rolling curves. (Compare *Plane Curves, Circle* using, Parameter Settings: $hh = -0.125$, $ii = 4$, and increase t-Resolution to 200, then choose Generalized Cycloids from the Action Menu.) Since the absolute value of the field increases along the field lines from the equator towards the poles, one cannot have solutions that almost follow the field lines while circling them in narrow loops, however one can approximate such behavior with the initial condition $v_x = 0.035, v_y = 0.035, v_z = 0.05$.

In a Plenary address on Dynamical Systems he gave at the 1998 International Congress of Mathematicains in Berlin, Jürgen Moser had an interesting discussion of the Störmer Problem that we reproduce below from Documenta Mathematica, Extra Volume, ICM 1998, pp. 381–402. (After the lecture, one of us approached Moser and showed him the visualization of the Störmer Problem in 3D-XplorMath. He appeared to be delighted by it, but said something looked not quite right to him, a remark that helped us eliminate a small bug!)

Here is the extract from Moser's lecture.

R.S.P. & H.K.

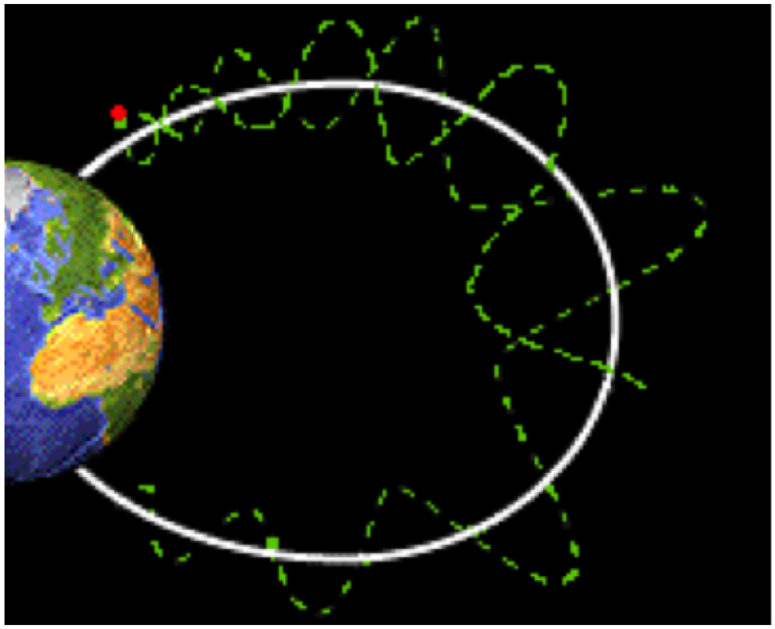
d) The Störmer Problem.

Another large scale confinement region is known in the magnetic field of the earth. With the advent in 1957 of satellites it was soon discovered that the earth was surrounded by (two) belts of charged particles caused by its magnetic field. Since the beginning of the century it was known that such charged particles were present above the atmosphere and were responsible for the aurora borealis (and australius). It was Störmer (incidentally president of the ICM 1936 in Oslo) who made calculations of the orbits of these charged particles moving in the magnetic field of the earth, which he modelled as a magnetic dipole field. This is an interesting nonlinear Hamiltonian system.

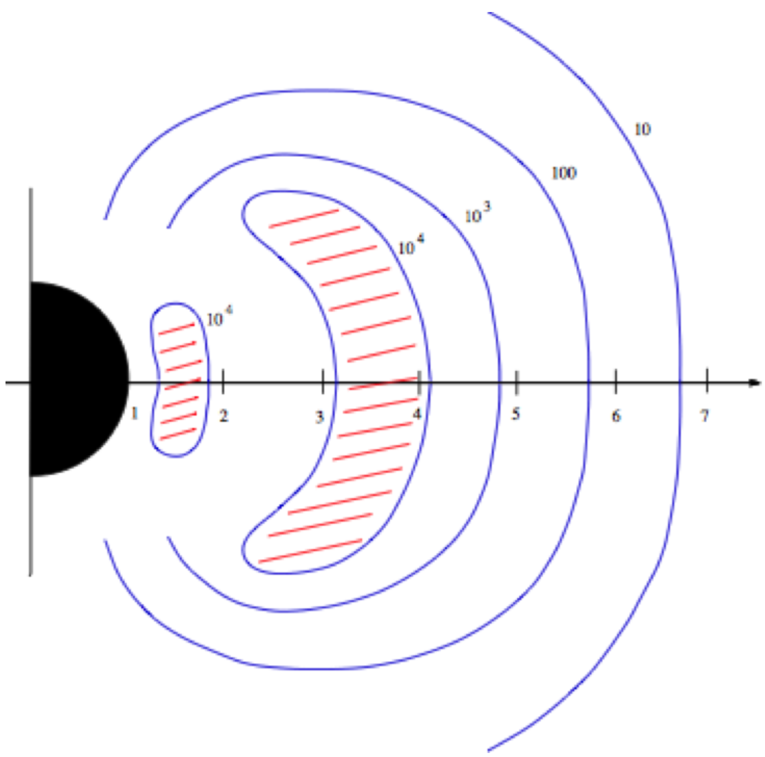
The satellite measurements led to the discovery of two regions surrounding the earth, the so-called van Allan belts, in which charged particles were trapped. It turns out that it is an example of a magnetic bottle to which the stability theory is applicable (M. Braun 1970).

It is interesting to realize the dimensions involved: For electrons, the “cyclotron radius” is of the order of a few kilometers and the corresponding periods of oscillation about one millionth of a second! The “bounce period” of travel from north pole to south pole and back is a fraction of a second.

In addition to these natural van Allan belts several artificial radiation belts have been made by the explosion of high-altitude nuclear bombs since 1958. Some of those so created belts had a lifetime up to several years—which shows the long stability of these experiments as well as the irresponsibility for carrying them out! Some 30 years ago these tests have been stopped.



Störmer Problem



Van Allen Belt