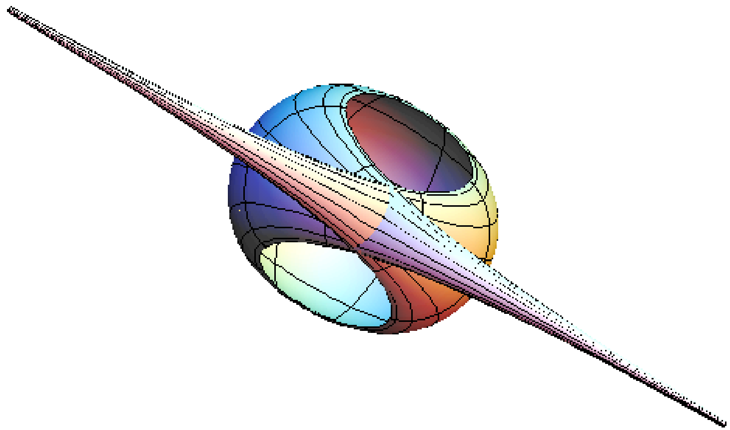
About the Sievert-Enneper Surface *



The Sievert-Enneper Surface

This surface stems from the PhD thesis of Sievert which he wrote under Enneper. It is a surface of constant Gauss curvature $K = 1$. Such surfaces are locally isometric to the unit sphere and are therefore also called *spherical surfaces*. Locally isometric means: if we make the Sievert-Enneper Surface from a thin sheet of metal (which should suggest that we can deform the surface by bending without stretching), we can take the surface by its tails and wrap it around the equator of the unit sphere; it will bend into the shape of the sphere, but it is not big enough to cover the whole sphere. Choose extreme values,

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* This file is from the 3D-XploreMath project. Please see http://rsp.math.brandeis.edu/3D-XplorMath/index.html

e.g. $aa = 0.05, aa = 30$; in the latter case large portions of the surface are already very spherically shaped, but the holes are always there. So one may wonder why one does not extend Sievert's surface beyond its boundary. A similar phenomenon can more easily be explained for those $K = 1$ surfaces of revolution which have their equator longer than that of the unit sphere (and which are obtained in 3DXM by setting $cc > 1$ in the Set Parameters Dialog). They, too, have holes instead of polar regions. In this case one can see from the formulas that the meridians are more and more strongly curved as their distance from the equator increases; the holes cannot be closed because the curvature of the meridians becomes ∞ at the rim of each hole.

The Sievert-Enneper Surface has even more in common with these $K = 1$ surfaces of revolution. As with them the parameter lines are principal curvature directions. This means that the surface is least strongly, respectively most strongly, curved in the directions of the two parameter lines through any point. On a surface of revolution these principal curvature lines are planar curves, the meridians and the latitudes. The infinitely long parameter lines of the Sievert-Enneper surface are also *planar curves*, and these planes all contain the y-axis, the line which is tangent to the two tails at infinity. (This is clear from the formulas below because the function Φ depends only on u and not on v.) The second family of parameter lines on the Sievert-Enneper Surface ($v = const$) looks planar also, but this is not the case. However, as aa becomes very large so that the surface looks very spherical, one observes that the two families of parameter lines converge to two orthogonal families of circles on the sphere. This explains why they look almost planar.

The formulas for the Sievert-Enneper Surface do not look very appealing. We write them with the help of three auxiliary functions:

$$\Phi(u) := -u/\sqrt{aa+1} + \arctan(\sqrt{aa+1} \cdot \tan u),$$
$$a(u,v) := 2/(aa + 1 - aa \sin^2 v \cos^2 u),$$
$$r(u,v) := a(u,v)\sqrt{(1 + 1/aa)(1 + aa \sin^2 u)} \sin v.$$

The Sievert-Enneper Surfaces:

$$x(u,v) := r(u,v) \cos \Phi(u),$$
$$y(u,v) := r(u,v) \sin \Phi(u),$$
$$z(u,v) := \frac{\log(\tan(v/2)) + (aa+1)a(u,v) \cos v}{\sqrt{aa}}.$$