Symmetries Of Elliptic Functions*

[The approach below to elliptic functions follows that given in section 3 of ”The Genus One Helicoid and the Minimal Surfaces that led to its Discovery”, by David Hoffman, Hermann Karcher, and Fusheng Wei, published in Global Analysis and Modern Mathematics, Publish or Perish Press, 1993. For convenience, the full text of section 3 (without diagrams) has been made an appendix to the chapter on the Conformal Map Category in the documentation of 3D-XplorMath.]

An elliptic function is a doubly periodic meromorphic function, $F(z)$, on the complex plane $\mathbb{C}$. The subgroup $\mathbb{L}$ of $\mathbb{C}$ consisting of the periods of F (the period lattice) is isomorphic to the direct sum of two copies of $\mathbb{Z}$, so that the quotient, $T = \mathbb{C}/\mathbb{L}$, is a torus with a conformal structure, i.e., a Riemann surface of genus one. Since F is well-defined on $\mathbb{C}/\mathbb{L}$, we may equally well consider it as a meromorphic function on the Riemann surface T.

It is well-known that the conformal equivalence class of such a complex torus can be described by a single complex number. If we choose two generators for $\mathbb{L}$ then, without changing the conformal class of $\mathbb{C}/\mathbb{L}$, we can rotate and scale the lattice so that one generator is the complex num-

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

ber 1, and the other, τ, then determines the conformal class of T. Moreover, τ_1 and τ_2 determine the same conformal class if and only if they are conjugate under $SL(2, Z)$.

The simplest elliptic functions are those defining a degree two map of T to the Riemann sphere. We will be concerned with four such functions, that we call JD, JE, JF, and WP. The first three are closely related to the classical Jacobi elliptic functions, but have normalizations that are better adapted to certain geometric purposes, and similarly WP is a version of the Weierstrass $\wp$-function, with a geometric normalization. Any of these four functions can be considered as the projection of a branched covering over the Riemann sphere with total space T, and as such it has four branch values, i.e., points of the Riemann sphere where the ramification index is two. For JD there is a complex number D such that these four branch values are $\{D, -D, 1/D, -1/D\}$. Similarly for JE and JF there are complex numbers E and F so that the branch values are $\{E, -E, 1/E, -1/E\}$ and $\{F, -F, 1/F, -1/F\}$ respectively, while for WP there is a complex number P such that the branch values are $\{P, -1/P, 0, \infty\}$. The cross-ratio, λ, of these branch values (in proper order) determines τ and likewise is determined by τ.

The branch values E, F, and P of JE, JF, and WP can be easily computed from the branch value D of JD (and hence from dd) using the following formulas:

$$E = (D-1)/(D+1), \quad F = -i(D-i)/(D+i),$$
$$P = i(D^2+1)/(D^2-1),$$

and we will use D as our preferred parameter for describing the conformal class of T. In 3D-XplorMath, D is related to the parameter dd (of the Set Parameter... dialog) by $D = \exp(dd)$, i.e., if $dd = a + ib$, the $D = \exp(a)\exp(ib)$. This is convenient, since if D lies on the unit circle (i.e., if dd is imaginary) then the torus is rectilinear, while if D has equal real and imaginary parts (i.e., if $b = \pi/4$) then the torus is rhombic. (The square torus being both rectilinear and rhombic, corresponds to $dd = i \cdot \pi/4$).

To completely specify an elliptic function in 3D-XplorMath, choose one of JD, JE, JF, or WP from the Conformal Map menu, and specify dd in the Set Parameter... dialog. (Choosing Elliptic Function from the Conformal map menu will give the default choices of JD and a square torus.)

When elliptic functions where first constructed by Jacobi and by Weierstrass these authors assumed that the lattice of the torus was given. On the other hand, in Algebraic Geometry, tori appeared as elliptic curves. In this representation the branch values of functions on the torus are given with the equation, while an integration of a holomorphic form (unique up to a multiplicative constant) is required to find the lattice. Therefore the relation between the period quotient τ (or rather its $SL(2,Z)$-orbit) and the cross ratio lambda of the four branch values has been well-studied. More recently, in Minimal Surface Theory, it was also more convenient to assume that the branch values of a degree two elliptic functions were given and that the periods had to be computed. Moreover, symmetries became more important than in the earlier studies.

Note that the four branch points of a degree two elliptic function (also called "two-division points", or Zweiteilungspunkte) form a half-period lattice. There are three involutions of the torus which permute these branch points; each of these involutions has again four fixed-points and these are all midpoints between the four branch points. Since each of the involutions permutes the branch points, it transforms the elliptic function by a Moebius transformation. In Minimal Surface Theory, period conditions could be solved without computations if those Moebius transformations were not arbitrary, but rather were isometric rotations of the Riemann sphere—see in the Surface Category the minimal surfaces by Riemann and those named Jd and Je. This suggested the following construction: As degree two MAPS from a torus ($T = \mathbb{C}/\mathbb{L}$) to a sphere, we have the natural quotient maps $T/-id$; these maps have four branch points, since the 180 degree rotations have four fixed points. To get well defined FUNCTIONS we have to choose three points and send them to $\{0, 1, \infty\}$. We choose these points from the midpoints between the branch points, and the different choices lead to different functions. The symmetries also determine the points that are sent to $\{-1, +i, -i\}$. In this way we get the most symmetric elliptic functions, and they are denoted JD, JE, JF. The program allows one to compare them with Jacobi's elliptic functions. The function $WP = JE * JF$ has a double zero, a double pole and the values $\{+i, -i\}$ on certain midpoints (diagonal ones in the case of rectangular tori). Up to an additive and a multiplicative constant it agrees with the Weierstrass $\wp$-function, but in our normalization it is the Gauss map of Riemann's minimal surface on each rectangular torus.

We compute the J-functions as follows. If one branch value is called +B, then the others are $\{-B, +1/B, -1/B\}$. Therefore the function satisfies the differential equations

$$(J')^2 = (J'(0))^2(J^4 + 1 - (B^2 + 1/B^2)J^2) = F(J),$$
$$J'' = (J'(0))^2(2J^3 - (B^2 + 1/B^2)J = F'(J)/2).$$

Numerically we solve this with a fourth order scheme that has the analytic continuation of the square root $J' = \sqrt{J'^2}$ built into it:

Let $J(0), J'(0)$ be given. Compute $J''(0) := F'(J(0))/2$ and, for small z,

$$J_m := J(0) + J'(0) \cdot z/2 + J''(0) \cdot z^2/8, \;\; J''_m := F'(J_m)/2,$$
$$J(z) := J(0) + J'(0) \cdot z + (J''(0) + 2 \cdot J''_m) \cdot z^2/6.$$

Finally let $J'(z)$ be that square root of $F(J(z))$ that is closer to $J'(0) + J''_m \cdot z$ (analytic continuation!). Repeat.

H.K.