Pinkall's flat tori in $\mathbb{S}^3$

1. Formulas

We can parametrize $\mathbb{S}^3$, considered as a submanifold of $\mathbb{C}^2$, by:

$$F(u, \alpha, v) = (\cos(\alpha)e^{iu}e^{iv}, \sin(\alpha)e^{iu}e^{-iv}),$$

where $u \in [0, 2\pi)$, $\alpha \in [0, \pi/2]$, and $v \in [0, \pi]$. We will get the Pinkall Tori first as flat tori in $\mathbb{S}^3$ by taking α to be a function of v, $\alpha := aa + bb \sin(\mathrm{ee}\, 2v)$. (The theory allows more general choices.)
Next we stereographically project $\mathbb{S}^3$ from

$$p = (\cos(cc \cdot \pi), 0, \sin(cc \cdot \pi), 0)$$

to get **conformal** images of the flat tori in $\mathbb{S}^3$. The lines $v = const$ are circles, the stereographic images of the Hopf circles $u \mapsto F(u, \alpha, v)$.
Finally, by morphing $0 \leq ff \leq 2\pi$, we can isometrically rotate $\mathbb{S}^3$ so that the Hopf circle $v = 0$ is the rotation axis. The stereographic image of this rotation is a conformal transformation of $\mathbb{R}^3 \cup \{\infty\}$ which "rotates" $\mathbb{R}^3$ around a circle on the pictured torus. In the case $aa = \pi/4$ we obtain for $ff = 0$

and $ff = \pi$ the same torus, but inside and outside interchanged. This is best viewed with the default 'Two Sided User Coloration'. It can be selected from a submenu of the Action Menu.

2. Background and Explanations

The tori which we usually see are, from the point of view of complex analysis, rectangular tori. This means: They have an orientation reversing symmetry and the set of fixed points of this symmetry has two components. The well known tori of revolution have isometric reflections with **two** circles as fixed point sets. Of course one tries to deform these tori to obtain nonrectangular ones. Obviously one can destroy the mirror symmetry, but this does not imply that one gets tori with a nonrectangular complex structure. The first proof, by Garcia, that one can embed all tori in $\mathbb{R}^3$ was nonconstructive and difficult. Pinkall's construction gives completely explicit flat tori in $\mathbb{S}^3$. They have a one parameter family of nonintersecting great circles on them which are parallel in the flat geometry of the torus. Stereographic projection from $\mathbb{S}^3$ to $\mathbb{R}^3$ gives conformal images of these flat tori.

The great circles $u \mapsto F(u, \alpha, v)$ are the orbits of the Hopf-action of $\mathbb{S}^1$ on $\mathbb{S}^3$, $(u,p) \mapsto e^{iu}p$. Each such "Hopf Fiber" lies in one of the parallel tori $\alpha =$ constant, and the great circles $\alpha \mapsto F(u,\alpha,v)$, meet these tori orthogonally, so that α measures the ditance between them. Taking $\alpha = \pi/4$ gives the "Clifford Torus" in $\mathbb{S}^3$, a minimal embedding of the square torus. For all α in $(0,\pi/4)$, the lengths of two orthogonal generators of the corresponding torus are $2\pi\cos(\alpha)$ and $2\pi\sin(\alpha)$.

The set of Hopf-Fibers with the natural distance in $\mathbb{S}^3$ is a metric space, isometric to the 2-sphere of radius $1/2$ (curvature 4). In fact, $(\alpha, 2v)$ are polar coordinates for this sphere. Pinkall observed that for any closed curve $(\alpha(s), \phi(s))$ on this sphere (with $\alpha(s)$ never equal to 0 or $\pi/2$) $(u,s) \mapsto (F(u,\alpha(s),v(s))$ gives an immersed torus. On these tori we still have the Hopf-Fibers, and since these are equidistant it follows that the metric is flat. Pinkall proved that the length and area of the curve in $\mathbb{S}^2$ determine the conformal structure of the torus, and that all conformal structures occur.

Observe that the usual tori of revolution in $\mathbb{R}^3$ are all rectangular, and most of the Pinkall tori shown by 3D-XplorMath are very different from these. The tori with aa $= \pi/4$ are all rhombic, because they can be rotated into themselves by 180° rotations (in $\mathbb{S}^3$, not $\mathbb{R}^3$) around any of the Hopf-Fibers on them. A cyclic morph with $0 \leq ff \leq 2\pi$ rotates around the circle $v = 0$. This rotation is an anti-involution of the torus with the circle as the (connected) fixed point set – only rhombic tori have such anti-involutions. (The square torus is rectangular and rhombic.)

The program takes $\alpha(v) := aa + bb \sin(\mathrm{ee}\, 2v)$ (with $ee = 3$ for the default image and $ee = 5$ for the default morph). These tori are shown in $\mathbb{R}^3$ by using the (conformal) stereographic projection of $\mathbb{S}^3 \setminus \{p\} \to \mathbb{R}^3$, where $p = (\cos(cc \cdot \pi), 0, \sin(cc \cdot \pi), 0)$. Morphing cc also gives conformal deformations of these tori.

H.K.