About Spherical Curves

In many ways there is a close analogy between planar Euclidean geometry and two-dimensional spherical geometry. In the ATO for spherical ellipses we translate the sum-of-distances definition from the plane to the sphere and use the same arguments as in the plane to construct points and tangents of the curve. Similarly, in the ATO about spherical cycloids, we roll spherical circles on spherical circles. Such analogies of course require basic notions which correspond to each other.

Lines and Triangles

Straight lines in the plane are the shortest connections between their points. On the sphere the shortest connections are great circle arcs that are not longer than half way around. A line cuts the plane into two congruent half-planes that are interchanged by the reflection in the line. Similarly, the sphere is cut by the plane of a great circle into two congruent half-spheres, and the reflection in the plane interchanges these two half-spheres. Therefore we speak of the reflection (of the sphere) in a great circle. These analogies are enough to translate the planar notion *straight line* to the spherical notion *great circle*. Three points A, B, C and three shortest connections of lengths a, b, c make a triangle — in the plane or on the sphere. The angles at the points (or vertices) are denoted α, β, γ. For the plane, the basic triangle formulas are close to the definition of sine and cosine:

Projection theorem:	$c = a \cdot \cos\beta + b \cdot \cos\alpha$
Sine theorem:	$b \cdot \sin\alpha = h_c = a \cdot \sin\beta$
Cosine theorem:	$c^2 = a^2 + b^2 - 2ab\cos\gamma$

Note that the more complicated third formula follows from the first two: Use the Sine theorem in the form $0 = b \cdot \sin\alpha - a \cdot \sin\beta$ and add the square of this to the square of the Projection theorem. Simplify with $\cos^2 + \sin^2 = 1$ and use the trigonometric identity $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta) = \cos(\pi-\gamma) = -\cos(\gamma)$ to obtain the Cosine theorem.

To derive similar formulas for spherical triangles, use geographic coordinates on the standard unit sphere, with the polar center at the north pole $C := (0, 0, 1)$. A point A at spherical distance b from C satisfies

$\langle A, C\rangle = \cos b$. Thus, after rotation into the x-z-plane, it has coordinates $A := (\sin b, 0, \cos b)$. A third point B at distance a from the pole C and such that the angle $\angle ACB$ equals γ has spherical polar coordinates $B := (\sin a \cos\gamma, \sin a \sin\gamma, \cos a)$. The spherical cosine formula follows by taking a scalar product:

$$\langle A, B\rangle = \cos c = \cos a \cos b + \sin a \sin b \cos\gamma.$$

The name is justified since a Taylor approximation up to second order gives the corresponding formula for the plane.

(For more details: believe that the graph of the function $x \longrightarrow \cos x$ lies above the graph of the quadratic function $x \longrightarrow 1 - x^2/2$ and *not* above any wider parabola $x \longrightarrow 1-x^2/(2+\epsilon)$. Therefore $1-x^2/2$ is called the quadratic Taylor approximation of cos near $x = 0$. We substitute this approximation for $x = a, x = b, x = c$, and similarly $\sin x \approx x$, in the spherical cosine formula and obtain: $\cos c \approx 1 - c^2/2 \approx (1-a^2/2)(1-b^2/2) + ab\cos\gamma$, or $c^2 \approx a^2+b^2-2ab\cos\gamma$, i.e. approaching the planar formula.)

For a more systematic derivation we use the reflection R which interchanges C, A and observe that $R(B)$ has the coordinates $R(B) := (\sin c \cos\alpha, \sin c \sin\alpha, \cos c)$.

But $R(B)$ can also be computed from the reflection matrix and the coordinates of B. Equating the two expressions gives three formulas between a, b, γ on one side and c, α on the other side. Of course these formulas hold for any permutation of A, B, C:

$$\begin{pmatrix} -\cos b & 0 & \sin b \\ 0 & 1 & 0 \\ \sin b & 0 & \cos b \end{pmatrix} \cdot \begin{pmatrix} \sin a \cos \gamma \\ \sin a \sin \gamma \\ \cos a \end{pmatrix} =$$

$$\begin{pmatrix} -\sin a \cos b \cos \gamma + \cos a \sin b \\ \sin a \sin \gamma \\ \cos a \cos b + \sin a \sin b \cos \gamma \end{pmatrix} = \begin{pmatrix} \sin c \cos \alpha \\ \sin c \sin \alpha \\ \cos c \end{pmatrix}.$$

We use for these formulas the same names as in the planar case since an even simpler Taylor approximation simplifies also the first two equations to their planar counterparts:

Projection theorem: $\sin c \cos \alpha =$

$$-\sin a \cos b \cos \gamma + \cos a \sin b$$

Sine theorem: $\sin c \cdot \sin \alpha = \sin a \cdot \sin \gamma$

Cosine thm: $\cos c = \cos a \cos b + \sin a \sin b \cos \gamma.$

A consequence of the first two theorems is the

Angle cosine: $\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c.$

Application: Platonic Solids

Two-dimensional spherical geometry captures certain aspects of three-dimensional Euclidean geometry. For example, if we project an *icosahedron* from its center to its circumsphere then the 20 triangular faces of the icosahedron are mapped to a tessellation of $\mathbb{S}^2$ by 20 equilateral triangles *whose angles are* $72°$ *because five triangles meet at every vertex.* From the angle cosine theorem we read off the edge-length σ of these triangles, with $\alpha = 2\pi/5$ we have for the

Icosahedron: $(\cos\alpha + \cos^2\alpha)/\sin^2\alpha = \cos\sigma$.

Given the above spherical tools this is a conceptually very simple construction.

Osculating Circles

At every point of a twice differentiable curve c on $\mathbb{S}^2$ one can determine its osculating circle: the parametrized circle that agrees with c up to the second derivative at that point. While it is easy to place a ruler next to a curve so that the ruler approximates a tangent line, one cannot so easily guess these best approximating circles. For all planar curves and space curves in 3DXM one can choose Osculating Circles from the Action Menu and one can believe that the resulting images show best approximating circles. In the case of spherical curves one observes that these osculating circles actually lie on the sphere. To understand this, consider the usual osculating circle in $\mathbb{R}^3$ and intersect its plane, the *osculating plane* of the curve c, with $\mathbb{S}^2$; this intersection circle is clearly a better approximation of the curve than any other circle in this plane and therefore it is the osculating circle. Although we cannot yet describe the *curvature* of a curve by a real valued function, we can already agree that, at each point, a space curve is curved as strongly as its osculating circle. We call the spherical radii of these circles the spherical curvature radii and we are ready to translate geometric constructions (with curves) from the plane to the sphere.

Parallel curves of a spherical curve c on $\mathbb{S}^2$. We define $\eta(t) := \dot{c}(t) \times c(t)/|\dot{c}(t)|$ as the oriented spherical unit normal of c. The parallel curve at spherical distance ϵ is then in complete analogy with the plane given as

$$\text{Parallel Curves on } \mathbb{S}^2: \; c_\epsilon(t) := \cos\epsilon \cdot c(t) + \sin\epsilon \cdot \eta(t).$$

It is easy to check that the curvature radii of c_ϵ are obtained by adding ϵ to the curvature radii of c — which is what our intuition expects of parallel curves.

Spherical Evolvents (also called involutes). For a physical realization of an evolvent attach a string segment to the curve and move the end point so that the string is always tangent to the curve, in the forward or in the backward direction. The Euclidean formula for the backwards evolvent is (assuming $|\dot{c}(t)| = 1$)

$$e(t) := c(t) - (t - t_0) \cdot \dot{c}(t), \; t \geq t_0.$$

A remarkable property of the evolvent is that $t - t_0$ is its curvature radius at $e(t)$.

It is easy to translate this construction to the sphere. The formula for the spherical evolvent is (assuming again $|\dot{c}(t)| = 1$)

$$e(t) := \cos(t - t_0) \cdot c(t) - \sin(t - t_0) \cdot \dot{c}(t).$$

A short computation shows that the spherical curvature radius at $e(t)$ is $t - t_0$, as in the plane. Also, it is true for the plane and for the sphere that the segment from $c(t)$ to $e(t)$ is orthogonal to $\dot{e}(t)$, i.e., this segment is the curvature radius of the evolvent at $e(t)$.

Spherical Evolutes. For any given (planar or) spherical curve c we call the curve of the (planar or) spherical midpoints of the osculating circles of c the *(planar or) spherical evolute* of c. In 3DXM this can best be seen in the demo for *Spherical Ellipses*. In the previous paragraph we have seen that, in the plane and on the sphere, the evolvent of the evolute of c is this given curve c. Thus, the natural translations of notions from the plane to the sphere continue to have natural properties.

What is Curvature?

More precisely, what real number should measure the size of the curvature at one point of the curve c, and which real valued function should describe the curvatures of c? For the plane, differential geometers have agreed to take the rotation speed of a unit normal of c as the quantitative size of its curvature. For example, the rotation speed of the unit normal n of a circle of radius r (use arc length parametrization) is $1/r$, since $c(t) = r \cdot (\cos(t/r), \sin(t/r)),\ |\dot{c}(t)| = 1$ and $n(t) = (\cos(t/r), \sin(t/r))$, hence $\dot{n}(t) = (1/r) \cdot \dot{c}(t)$. Although this is a good reason for taking $1/r$ as the curvature of a circle of radius r in the plane, the argument does not carry over to $\mathbb{S}^2$, since: *What is the spherical rotation speed of the spherical normal?* Of course we could also call on the sphere $1/$*curvature radius* the curvature of the curve. This is not a good idea on $\mathbb{S}^2$ since circles of radius $\pi/2$ are great circles, i.e., shortest connections, and we would expect them to have curvature 0. Fortunately, there is for the plane another good reason for taking $1/r$ as "the" curvature, and this time the corresponding computation can be repeated on $\mathbb{S}^2$. If we imagine a family of parallel curves then it looks as if the length grows faster if the curvature is larger.

We can make this intuition more precise with a computation. First, in the plane:

$$c_\epsilon(t) := c(t) + \epsilon \cdot n(t),\ \{\dot c(t), n(t)\} \text{ orthonormal}$$
$$\dot c_\epsilon(t) = \dot c(t) + \epsilon \cdot \dot n(t),\ \ \dot n(t) = \kappa(t) \cdot \dot c(t)$$
$$\frac{d}{d\epsilon}|\dot c_\epsilon(t)|_{\epsilon=0}/|\dot c(t)| = \kappa(t).$$

Here, the second line defines the curvature as the rotation speed of the normal and the third line says *that this curvature function can also be computed as the change of length of tangent vectors in a parallel family of curves.* Of course we can do the same computation as in line three for spherical curves:

$$
\begin{aligned}
&c_\epsilon(t) := \cos\epsilon \cdot c(t) + \sin\epsilon \cdot \eta(t)\\
&\dot c_\epsilon(t) = \cos\epsilon \cdot \dot c(t) + \sin\epsilon \cdot \dot\eta(t)\\
&\dot\eta(t) = \ddot c(t) \times c(t)\\
&|\dot c_\epsilon(t)|/|\dot c(t)| = \langle \dot c_\epsilon(t), \dot c(t)\rangle / \langle \dot c(t), \dot c(t)\rangle =\\
&\cos\epsilon + \sin\epsilon \langle \dot\eta(t), \dot c(t)\rangle / \langle \dot c(t), \dot c(t)\rangle\\
&\frac{d}{d\epsilon}|\dot c_\epsilon(t)|_{\epsilon=0}/|\dot c(t)| = -\langle \eta(t), \ddot c(t)\rangle / \langle \dot c(t), \dot c(t)\rangle.
\end{aligned}
$$

Before we take this as the definition of spherical curvature for spherical curves we check which function of the radius we get for circles of spherical radius r:

$$
\begin{aligned}
&c_r(t) = (\sin r \cos t, \sin r \sin t, \cos r)\\
&\eta(t) = \frac{d}{dr} c_r(t) = (\cos r \cos t, \cos r \sin t, -\sin r)\\
&\ddot c_r(t) = -(\sin r \cos t, \sin r \sin t, 0), \qquad \text{finally:}\\
&-\langle \eta(t), \ddot c(t)\rangle / \langle \dot c(t), \dot c(t)\rangle = \frac{\sin r \cos r}{\sin^2 r} = \cot r.
\end{aligned}
$$

This is a satisfying answer, since $\cot r$ behaves like $1/r$ for small r and $\cot(r = \pi/2) = 0$ as we expect for great circles. Now we are ready for the definition and we remark that the historical name is *geodesic curvature*, not the more naive spherical curvature which we used above.

Definition. The geodesic curvature $\kappa_g(t)$ of a spherical curve $c(t)$ with spherical unit normal $\eta(t)$ is

$$\kappa_g(t) := -\langle \eta(t), \ddot{c}(t)\rangle / \langle \dot{c}(t), \dot{c}(t)\rangle.$$

The Spherical Frenet Equation

Finally we observe that for a unit speed spherical curve c we have the following natural orthonormal frame along the curve:

$$(e_1(t), e_2(t), e_3(t)) := (\dot{c}(t), c(t), \eta(t)),$$

and the geodesic curvature controls the derivative of this frame via the following *spherical Frenet equation*:

$$\begin{aligned}
\frac{d}{dt}\dot{c}(t) &= \qquad\qquad\quad -1 \cdot c(t) - \kappa_g(t) \cdot \eta(t) \\
\frac{d}{dt}c(t) &= \quad +1 \cdot \dot{c}(t) \\
\frac{d}{dt}\eta(t) &= +\kappa_g(t) \cdot \dot{c}(t)
\end{aligned}$$

Observe that the coefficient matrix

$$\begin{pmatrix} 0 & -1 & -\kappa_g \\ 1 & 0 & 0 \\ \kappa_g & 0 & 0 \end{pmatrix}$$

is skew symmetric. This fact implies that any solution $(e(t), f(t), g(t))$ with *orthonormal* initial conditions stays orthonormal. This says that $t \to f(t)$ is a spherical curve parametrized by arclength (namely: $|\dot{f}(t)| = |e(t)| = 1$). Moreover $g(t)$ is orthogonal to $f(t), \dot{f}(t)$ and therefore the spherical unit normal of f. The third Frenet equation says that the given function $\kappa_g(t)$ (because of $\dot{g}(t) = \kappa_g(t) \cdot e(t)$) is indeed the geodesic curvature of the curve $t \to f(t)$: to any given $\kappa_g(t)$ we have found a curve with that geodesic curvature.

We repeat: from elementary distance and triangle geometry to the differential geometry of curves we have explained a very close analogy between the Euclidean plane and the sphere. The 3DXM demos try to emphasize this.

HK.