Cubic Curves*

Real Cubic Curves in $\mathbb{R}^2$ were studied extensively by Newton. Later these curves were considered as the real points of complex curves in $\mathbb{C}^2$. If they do not have double points they can be parametrized by additive groups. This means that the points on these curves can be added. Surprisingly this addition is a *geometric addition*, i.e. the sum $P+Q$ can be geometrically constructed from P, Q and the curve. In the case of cubic curves we have:

$$P+Q+R=0 \quad \Leftrightarrow$$
$$\text{A straight line intersects the curve in } P, Q, R\ .$$

In 3D-XplorMath are the following examples:

Cubic Polynomial Graph, $x(t)=t, y(t)=x(t)^3+aa\cdot x(t)$,

Cuspidal Cubic, $x(t)=3t^2/(4aa), y(t)=t(t^2+bb)/(4aa^2)$,

Cubic Rational Graph I, $x(t)=\tan(t/2)/aa,\ y(t)=\sin(t)$,

Cubic Rat'l Graph II, $x(t)=\tanh(t/2)/aa,\ y(t)=\sinh(t)$,

Elliptic Cubic, $x+1/x-aa\cdot(y-1/y)=ff$ (implicit),

Folium, $[x,\ y]=aa[(t^2-t^3),\ (t-2t^2+t^3)]/(1-3t-3t^2)$,

Nodal Cubic, $x=1-t^2,\ y=((1-t^2)+bb)\cdot t$.

The last two of these are cubics with one double point.

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

Their points do not form a group, therefore their Action Menu has only the Standard Actions for plane curves.

All the others are parametrized by a 1-dim. Abelian group and the curves are shown with a default demo explaining the geometric addition.
If we intersect a Cubic Polynomial Graph without quadratic term with a line, then the x-coordinates of the intersection points are always roots of a polynomial without quadratic term. In other words: these three x-coordinates add up to 0., the *geometric addition* is the standard addition on the x-axis.
The Cuspidal Cubic is also parametrized by $\mathbb{R}$ (or $\mathbb{C}$) and a simple computation shows: if $1/t_1 + 1/t_2 + 1/t_3 = 0$ then the three points $[x(t_i), y(t_i)]$ lie on a straight line. And dually, if $t_1 + t_2 + t_3 = 0$ then the tangents at the three points $[x(t_i), y(t_i)]$ pass through one point. Again, the addition has a simple geometric interpretation that allows to construct, if two points and the curve are given, their sum.
The first Rational Cubic Graph is parametrized by a circle $\mathbb{S}^1$ (we have to add the infinite point $(\infty, 0)$). The demo that comes with the curve shows how the sum point can be constructed by intersecting lines. The Action Menu offers a second demo that shows how addition on the parametrizing circle and on the curve are the same.
The second Rational Cubic Graph would not be here if we could visualize these curves over the complex Numbers. Over $\mathbb{C}$ one can think of this curve as the group $\mathbb{S}^1 \oplus \mathbb{R}$, a cylinder. The first rational graph visualizes the equator

circle, the second one visualizes the generator through the neutral element plus the opposite generator: two copies of $\mathbb{R}$ (and a double point at ∞).
The Elliptic Cubic is parametrized by a pair of so called *Elliptic Functions.* Such functions can be viewed either as doubly periodic functions from $\mathbb{C}$ to $\mathbb{S}^1$ or as functions defined on some torus. For more details see the text *Elliptic Functions.*
The addition on elliptic curves can be compared with addition on the circle. The formulas for trigonometric functions

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

show that

$$(x_1, y_1) \oplus (x_2, y_2) := (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2)$$

gives the addition of points $(x_1, y_1), (x_2, y_2) \in \mathbb{S}^1$. Notice that the rational points (i.e. the pythagorean triples) are a subgroup. The elliptic functions have analogous functional equations which are similarly the basis for addition formulas for points on elliptic cubics. This is explained in the text *Geometric Addition on Cubic Curves.*