About Minimal Surfaces

Discovery and physical interpretation.

The question *which surfaces locally minimize area* led Lagrange in 1760 to the minimal surface equation for graphs. By 1765 Meusnier had found that a geometric interpretation of this equation is: *the mean curvature of the surface vanishes.* He discovered that the catenoid and the helicoid are nonplanar examples. It took until 1835 for the next examples to appear, discovered by Scherk; his doubly periodic surface is a graph over the black squares of a checkerboard tesselation of the plane and his singly periodic surface is nowadays viewed as a desingularization of two orthogonally intersecting planes. In the following years complex analysis developed and by 1865 many examples were known through the efforts of Riemann, Weierstraß, Enneper and in particular Schwarz.

Also in that period Plateau had made careful experiments with soap films. He convinced people that soap films were a perfect physical realization of minimal surfaces, and he convinced mathematicians that they should solve *Plateau's Problem*, i.e. prove that every continuous injective closed curve in $\mathbb{R}^3$ spans a minimal surface. This problem was solved in 1932 by Douglas and independently by Rado. On the way to this solution mathematicians had learnt a lot about nonlinear elliptic partial differential equations. In particular the importance of the *maximum principle* had become clear, it implies for example that every compact minimal surface is contained in the convex hull of its boundary and that boundary value problems are well posed for the minimal surface equation.

On the other hand, although the Cauchy-Kowalewski theorem allows to solve locally initial value problems with analytic data, there is no continuous dependence on the data and no hope to obtain complete immersed examples with this method. — But, already Weierstraß had established the close connection of minimal surfaces with complex analysis. In particular: the spherical Gauss map composed with stereographic projection is locally a holomorphic function $G : M^2 \to \mathbb{C}$. In terms of the $90°$ rotation of each tangent space of the minimal surface M^2 holomorphicity has the following intuitive interpretation: for each $v \in TM^2$ we have the following version of the

Cauchy-Riemann equations

$$dG(\mathrm{Rot}(90^\circ) \circ v) = i \cdot dG(v).$$

Moreover, the three component functions of the immersion $(F^1, F^2, F^3) : M^2 \to \mathbb{R}^3$ are, locally, the real parts of holomorphic functions because the differential forms $\omega^j := -dF^j \circ \mathrm{Rot}(90^\circ)$ are closed for surfaces with mean curvature zero (Meusnier's above interpretation of "minimal"). This fact establishes the Weierstraß representation:

> Let G be the holomorphic Gauss map and $dh := dF^3 - i \cdot dF^3 \circ \mathrm{Rot}(90^\circ)$ the (holomorphic) complexification of the differential of the height function F^3 then

$$(F^1, F^2, F^3) = \mathrm{Re} \int \left(\frac{1}{2}(\frac{1}{G} - G), \frac{i}{2}(\frac{1}{G} + G), 1 \right) dh.$$

The examples of the second half of the 19$^{\mathrm{th}}$ century were made with this representation. But results, achieved by 1960 by Huber and Osserman show, that *all* minimal surfaces of a certain kind can be obtained by a global application of this representation, namely:

Complete, immersed minimal surfaces of finite total curvature can be conformally compactified by closing finitely many punctures; moreover, the Weierstraß data G, dh extend *meromorphically* to this *compact Riemann surface.*

The wealth of examples, discovered since about 1980, rely on this theorem. To understand these examples better we note the first and second fundamental forms (Riemannian metric and, if $|v| = 1$, normal curvature)

$$I(v,v) = \frac{1}{4}(\frac{1}{|G|} + |G|)^2 |dh(v)|^2$$
$$II(v,v) = \mathrm{Re}\frac{dG(v)}{G} dh(v).$$

The points p on the Riemann surface which are poles of dh do not correspond to points on the minimal surface. Every (differentiable) curve which runs into such a *puncture* p has infinite length on the minimal surface. The same is true if G has a zero or pole of higher order than the vanishing order of dh. If these orders are the same then we simply have a point with vertical normal on the minimal surface. And at points where the vanishing order of dh is larger, the metric becomes singular and the minimal surface has a so called *branch point*, it is no longer an immersion.

Visualization of minimal surfaces.

The Weierstraß representation allows to write down a number of simple minimal surfaces which can be visualized like any other surface for which an explicit parametrization is given. Our parameter lines come from polar coordinates with centers $\{0, \infty\}$ or $\{1, +1\}$. Note that the zeros and poles of G, dh fit together so that no branch points occur and so that the minimal surfaces are complete on the punctured spheres mentioned in each case. The surfaces are of finite total curvature, since the Gauss map is meromorphic, i.e., its image covers the Riemann sphere a finite number of times.

First Examples,
defined on $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$ or $\mathbb{S}^2 \setminus \{1, -1\}$:

Enneper Surface:

$$z \in \mathbb{C},\ G(z) := z,\ dh := zdz$$

Polynomial Enneper:

$$z \in \mathbb{C},\ G(z) := P(z),\ dh := P(z)dz$$

Rational Enneper:

$$z \in \mathbb{C},\ G(z) := P(z)/Q(z),\ dh := P(z)Q(z)dz$$

P and Q are polynomials without common zeros.

Vertical Catenoid:

$$z \in \mathbb{C} \setminus \{0\},\ G(z) := z,\ dh := dz/z,$$
$$\text{or } G(z) := 1/z$$

Helicoid:

$$z \in \mathbb{C},\ G(z) := \exp(z),\ dh := \mathbf{i}dz = \mathbf{i}\frac{dG}{G}$$

Helicoid:

$$z \in \mathbb{C} \setminus \{0\},\ G(z) := z,\ dh := \mathbf{i}dz/z$$

Planar to Enneper:

$$z \in \mathbb{C} \setminus \{0\},\ G(z) := z^{k+1},\ dh := z^{k-1}dz$$

Wavy Catenoid:

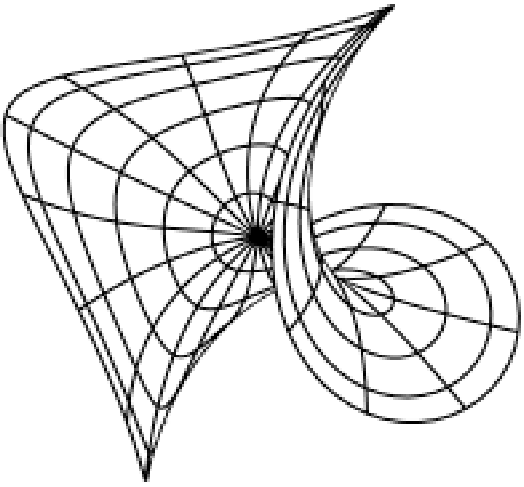
$$z \in \mathbb{C} \setminus \{0\},\ G(z) := (1 + \epsilon \cdot z^k)/z,\ dh := G(z)dz$$

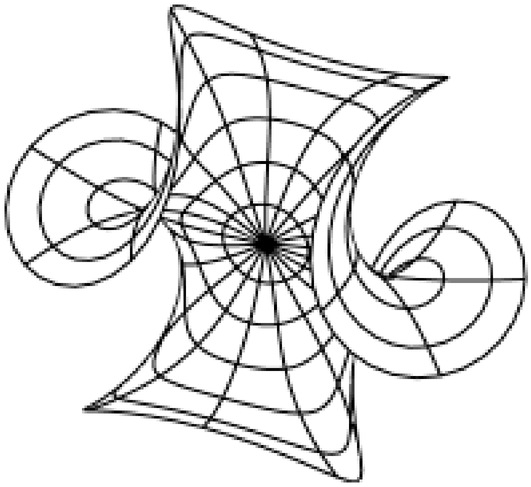
Wavy Plane:

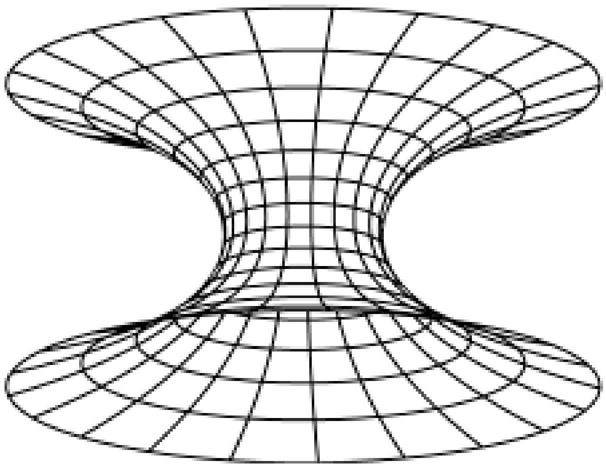
$$z \in \mathbb{C} \setminus \{0\},\ G(z) := z,\ dh := dz$$

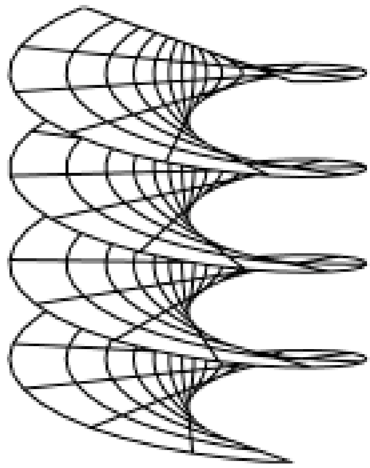
Horizontal Catenoid:

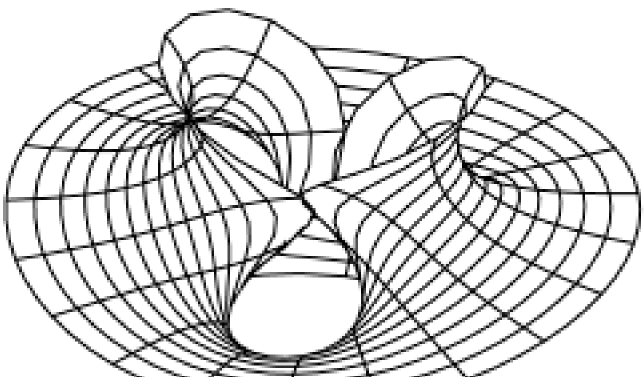
$$z \in \mathbb{S}^2 \setminus \{1, -1\},\ G(z) := z,\ dh := \frac{dz/z}{(z - 1/z)^2}.$$

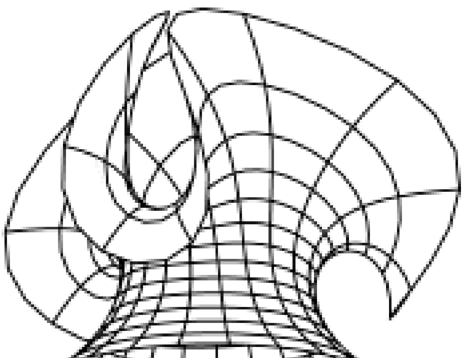


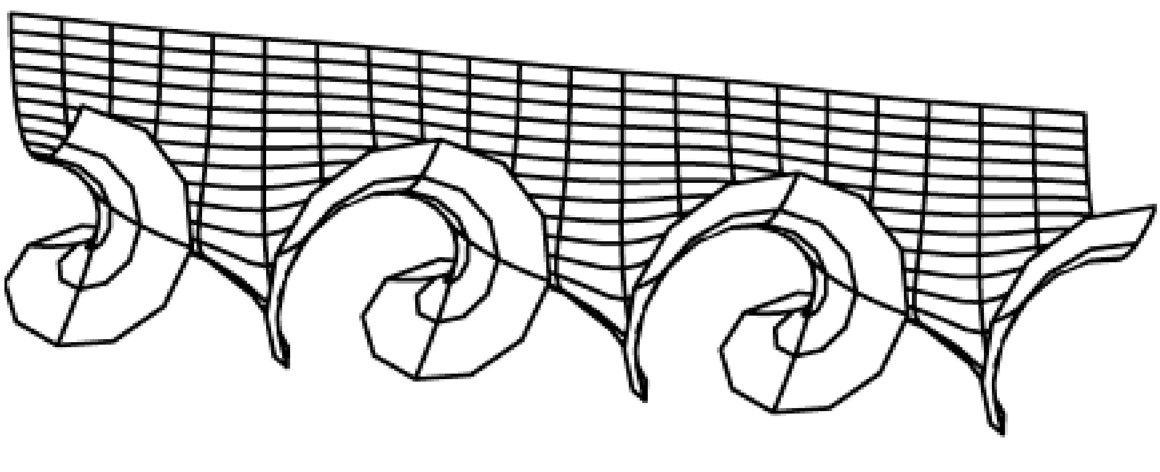












All of these simple minimal surfaces have *symmetries*: (i) straight lines on a minimal surface allow 180° rotations of the minimal surface into itsself, and (ii) planar geodesics on a minimal surface allow reflection (in the plane of the geodesic) of the minimal surface into itsself. Since these symmetries become more important for understanding more complicated surfaces one should learn how to recognize them. The straight lines are geodesics with normal curvature $II(c', c') = 0$ or $dG(c')/G \cdot dh(c') \in \mathbf{i} \cdot \mathbb{R}$. In the present context we recognize geodesics as fixed point sets of isometric involutions. The formula for the first fundamental form is so simple that one can easily see in all of these examples that the expected symmetry indeed does not change the Riemannian arc length of curves. To recognize the planar geodesics note that a geodesic on a surface is planar if it is also a principal curvature line; in addition to seeing it as the fixed point set of a length preserving involution we therefore only need to check $dG(c')/G \cdot dh(c') \in \mathbb{R}$, which is also easy in these examples.

In 3D-XplorMath one can easily change (in the Settings Menu) the range of the parametrization and also the symmetry of the surface. We recommend that the surfaces are looked at from far away when a large range for the parametrization is chosen. We also recommend to look at the default morphs of WavyEnneper and WavyCatenoid since it is quite surprising how suddenly the perturbation becomes visible. This should be taken as an illustration that the initial value problem for minimal surfaces is highly unstable, it is *ill posed* and no numerical solution is possible.

More complicated spherical examples.

The sudden increase of the interest in minimal surfaces after 1980 was largely caused by the discovery of a quite unexpected embedded finite total curvature minimal surface by Costa with embeddedness discovered and proved by Hoffman-Meeks. We are not yet close to such an example because of the following

Theorem of Lopez-Ros. An embedded, minimal, finite total curvature punctured sphere is a plane or a catenoid.

To practise using the Weierstraß representation we therefore have to be content with a few immersed punctured spheres. We want to learn *how to see the Gauss map* when one looks at the picture of such a minimal surface. The main fact to use is: a meromorphic function on a compact Riemann surface is determined up to a constant factor by its zeros and poles. In the case of the Jorge-Meeks k-noids one clearly sees a k-punctured sphere with a horizontal symmetry plane. One observes only two points with vertical normal, one up, one down. The qualitative behaviour of the Gauss map along the horizontal symmetry line suggests a mapping degree $k-1$. This leaves no choice but $G(z) = z^{k-1}$. If we look back at the very simple examples then we can observe that, at a catenoid like puncture, either Gdh or dh/G has precisely a double pole. This determines the dh below up to a constant factor.

The next two examples, the 4-noid with orthogonal ends of different size, and the double Enneper, have a quite different appearance, but they have the same Gauss map. The vertical points are symmetric with respect to the origin and symmetric with respect to the unit circle, and the degree of the Gauss map is three; this determines the Blaschke product expression below. In the case of the 4-noid we need to create the four catenoid ends with double poles of dh and we need to compensate the simple zeros and poles of G by simple poles of dh; then, if we also treat zero

and infinity symmetrically, the expression below is forced. In the case of the double Enneper surface we just need to compensate the simple zeros and poles of G (outside $0, \infty$); symmetric treatment of $0, \infty$ gives the dh below (except for a constant factor).
The last example illustrates in which way attempted counter examples to the Lopez-Ros theorem fail. A residue computation for the Weierstraß integrands shows that closed curves around the punctures ± 1 on the sphere are *not* closed curves on the minimal surface, if we want all limit normals to be vertical. It is easy to close this so called *period* when one allows tilted catenoid ends, but, as one decreases the tilt, the distance between the half catenoids increases, and they intersect the planar middle end if one computes the surface far enough towards the punctures.

The k-noids of Jorge-Meeks:

$z \in \mathbb{S}^2 \setminus \{e^{2\pi \mathbf{i} \cdot l/k};\ 0 \le l < k\}$,

$G(z) := z^{k-1}, \quad dh := (z^k + z^{-k} - 2)^{-1} \cdot dz/z.$

4-noids with two different orthogonal ends:

$z \in \mathbb{C} \setminus \{0, -1, +1\}, \quad G(z) := z \cdot \frac{z-r}{1-rz} \cdot \frac{z+r}{1+rz},$

$dh := \left(1 - \frac{z^2+z^{-2}}{r^2+r^{-2}}\right) \cdot (z^2 - z^{-2})^{-2} \cdot dz/z.$

Two Enneper ends joined by a catenoidal neck:

$z \in \mathbb{C} \setminus \{0\}$,

$G(z) := z \cdot \frac{z-r}{1-rz} \cdot \frac{z+r}{1+rz}, \quad dh := \left(1 - \frac{z^2+z^{-2}}{r^2+r^{-2}}\right) \cdot dz/z.$

Three punctures, period closes for tilted ends:

$z \in \mathbb{C} \setminus \{-1, +1\}$,

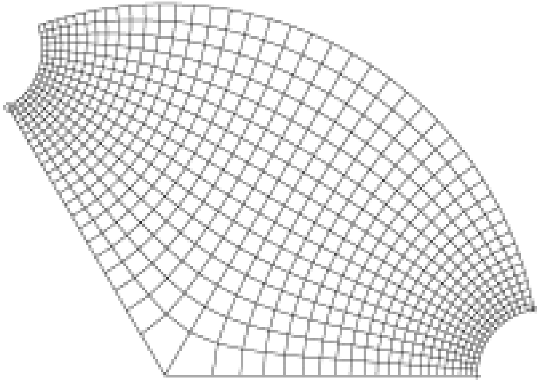
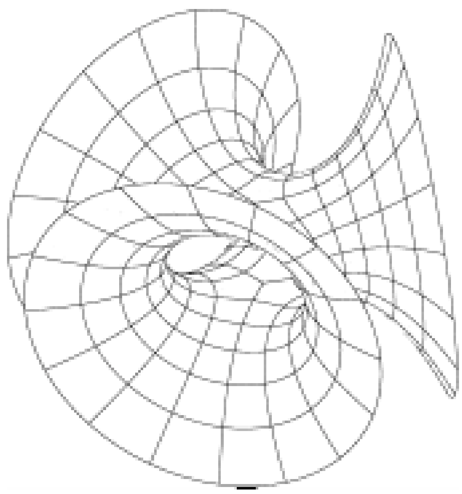
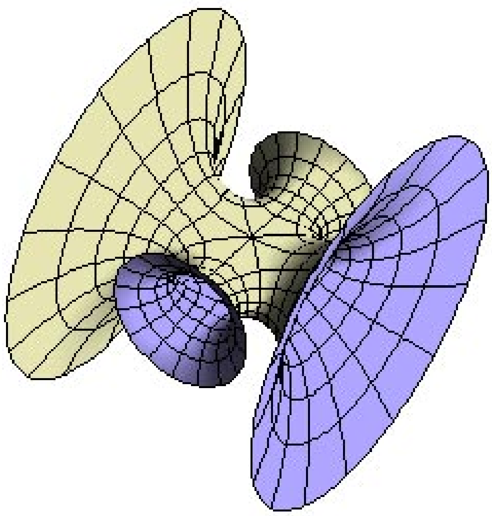
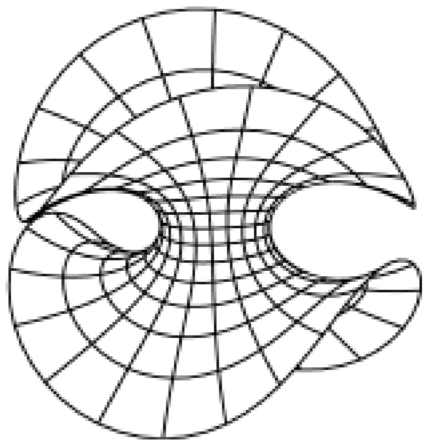
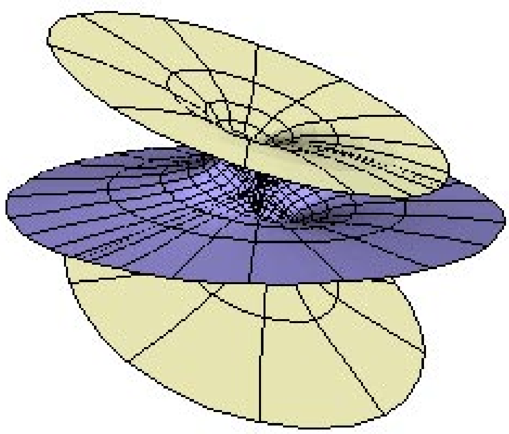
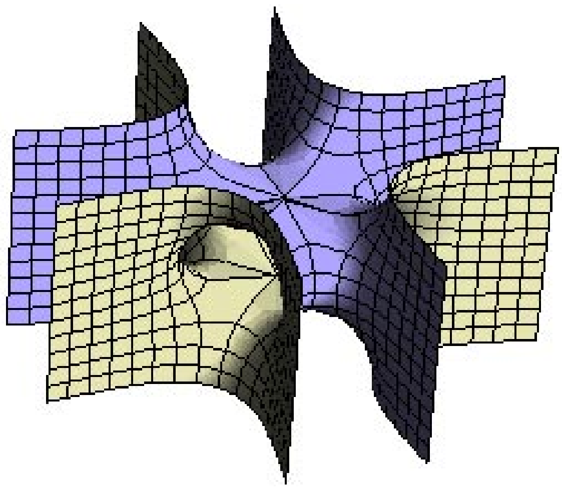
$G(z) := \rho(z^2 - r^2), \quad dh := \frac{z^2-r^2}{(z^2-1)^2} dz.$

Observe that the zeros and poles of the Gauss map which are not in the list of punctures are compensated by zeros of dh. At the embedded ends, Gdh or dh/G have a double pole and at the Enneper ends they have higher order poles. — In this list we do not have simple poles of Gdh and dh/G. If this happens then the Weierstraß integral behaves similar to $\int dz/z$: the unit disk, punctured at 0, is mapped by log to an infinite number of half strips parallel to the negative real axis and of width 2π. Similarly, the Weierstraß integral produces *simply periodic embedded minimal surfaces* parametrized by punctured spheres.

Generalized Scherk Saddle Towers:

$z \in \mathbb{S}^2 \setminus \{e^{\pm\phi} \cdot e^{2\pi \mathbf{i} \cdot l/k};\ 0 \le l < k\}$,

$G(z) := z^{k-1}, \quad dh := (z^k + z^{-k} - 2\cos k\phi)^{-1} \cdot dz/z.$

As in the simpler examples, observe that the symmetry lines can be seen from the Weierstraß data. We also note that at this point an important decision has to be made. If one represents the surfaces, as in all our examples, with parameter lines then each surface requires a special effort so that the parameter lines on the one hand support the complex analytic background of the minimal surface and on the other hand suggest correctly how one should imagine how the surface extends beyond what the picture shows. In 3D-XplorMath this individual approach has been taken. The other option is to spend considerably more general effort by writing software which will create a suitable triangulation of the domain. David and Jim Hoffman have such a program running. It requires much less individual work to compute another minimal surface but it is harder to illustrate the complex analysis background of the computed minimal surface.

The family of singly periodic embedded minimal surfaces which resemble the above generalized Scherk Saddle towers is much larger than the above explicit formulas suggest. So far we have only talked about the real part of the Weierstraß integral. In fact, a

1-parameter ("associate") family of isometric (and in general not congruent) minimal surfaces are given by this integral because dh can be changed by the factor $\exp(-2\pi\mathbf{i}\varphi)$. In particular, the imaginary part of the Weierstraß integral is the "conjugate" minimal surface. In the case of the generalized Scherk saddle towers we have that the conjugate minimal immersion maps the unit disk (with the punctures on the boundary) to a graph over a convex polygon; its edge lengths all agree. The minimal graph has over each edge the boundary value $+\infty$ or $-\infty$, alternatingly. — Jenkins-Serrin proved the converse: every such infinite boundary value problem has a graph solution, a minimal disk whose conjugate minimal surface is the fundamental piece of an embedded singly periodic saddle tower.

Having seen a good collection of minimal surfaces parametrized by punctured spheres we now turn to minimal surfaces parametrized by other Riemann surfaces.

H.K.