Plane Curves*

in 3D-XplorMath, a Visualization Program

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The Circle*

$$x = aa\cos(t),\ y = aa\sin(t),\ 0 \le t \le 2\pi$$

3DXM - SUGGESTION: Select from the Action Menu *Show Generalized Cycloid* and vary in the Settings Menu, entry: *Set Parameters*, the (integer) ratio between the radius aa and the rolling radius hh.
The length of the drawing stick is $ii*$rolling radius.

The circle is the simplest and best known closed curve in the plane. The default image shows the circle together with the theorem of Thales about right angled triangles. Other properties of the circle are also known since over 2000 years. In fact, many of the plane curves that have individual names were already considered (and named) by the ancient Greeks, and a large class of these can be obtained by rolling one *circle* on the inside or the outside of some other *circle*. The Greeks were interested in rolling constructions because it was their main tool for describing the motions of the planets (Ptolemy). The following curves from the Plane Curve menu can be obtained by rolling constructions:

Cycloid, Ellipse, Astroid, Deltoid, Cardioid, Limaçon, Nephroid, Epi- and Hypocycloids.

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Not all geometric properties of these curves follow easily from their definition as rolling curve, but in some cases the connection with complex functions (Conformal Category) does.

Cycloids arise by rolling a circle on a straight line. The parametric equations code for such a cycloid is

$P.x := aa \cdot t - bb \sin(t)$
$P.y := aa - bb \sin(t),\ aa = bb.$

Cycloids have other cycloids of the same size as evolute (Action Menu: "Show Osculating Circles with Normals"). This fact is responsible for Huyghen's cycloid pendulum to have a period independent of the amplitude of the oscillation.

Ellipses are obtained if *inside* a circle of radius aa another circle of radius $r = hh = 0.5aa$ rolls and then traces a curve with a radial stick of length $R = ii \cdot r$. The parametric equations for such an ellipse is

$P.x := (R + r) \cos(t)$
$P.y := (R - r) \sin(t).$

In the visualization of the complex map $z \to z + 1/z$ in Polar Coordinates the image of the circle of Radius R is such an ellipse with $r = 1/R$.

Astroids are obtained if *inside* a circle of radius aa another circle of radius $r = hh = 0.25aa$ rolls and then traces a curve with a radial stick of length $R = ii \cdot r = r$. Parametric equations for such Astroids are

$P.x := (aa - r)\cos(t) + R\cos(4t)$
$P.y := (aa - r)\sin(t) - R\sin(4t).$

Astroids can also be obtained by rolling the *larger* circle of radius r = hh = 0.75aa (put gg = 0 in this case). Another geometric construction of the Astroids uses the fact that the length of the segment of each tangent between the x-axis and the y-axis has **constant** length. — Try $hh := aa/3$ to obtain a **Deltoid**.

Cardioids and Limaçons are obtained if *outside* a circle of radius aa another circle of radius $r = hh = -aa$ rolls and then traces a curve with a radial stick of length $R = ii \cdot r, ii = 1$ for the Cardioids, $ii > 1$ for the Limaçons. Parametric equations for Cardioids and Limaçons are

$P.x := (aa + r)\cos(t) + R\cos(2t)$
$P.y := (aa + r)\sin(t) + R\sin(2t).$

The Cardioids and Limaçons can also be obtained by rolling the larger circle of radius $r = hh = +2aa$; now $ii < 1$ for the Limaçons. Note that the fixed circle is *inside* the larger rolling circle.
The evolute of the Cardioid (Action Menu: *Show Osculating Circles with Normals*) is a smaller Cardioid. The image of the unit circle unter the complex map $z \to w = (z^2 + 2z)$ is a Cardioid; images of larger circles are Limaçons. Inverses $z \to 1/w(z)$ of Limaçons are figure-eight shaped, one of them is a Lemniscate.

Nephroids are generated by rolling a circle of one radius outside of a second circle of twice the radius, as the program demonstrates. With $R = 3r$ we thus have the parametrization

$P.x := R\cos(t) + r\cos(3t)$
$P.y := R\sin(t) + r\sin(3t)$.

As with Cardioids and Limaçons one can also make the radius for the drawing stick shorter or longer: After selecting *Circle* set the parameters $aa = 1, hh = -0.5, ii = 1$ for the Nephroid and $ii > 1$ for its looping relatives. – Pick in the Action Menu: *Show Osculating Circles with Normals*. The Normals envelope a smaller Nephroid.

The complex map $z \to z^3 + 3z$ maps the unit circle to such a Nephroid. To see this, in the Conformal Map Category, select $z \to z^{ee} + ee \cdot z$ from the Conformal Map Menu, then choose Set Parameters from the Settings Menu and put $ee = 3$.

Archimedes' Angle Trisection. A demo of this construction can be selected from the Action Menu.

Circle Involute Gear. Another demo from the Action Menu. Involute Gear is used for heavy machinery because of the following two advantages: If one wheel rotates with constant angular velocity then so does the other, thus avoiding vibrations. If the teeth become thiner by usage, the axes can be moved closer to each other.

H.K.

The Ellipse*

$$x(t) = aa\cos(t),\ y(t) = bb\sin(t),\ 0 \le t \le 2\pi$$

3DXM-SUGGESTION:
Select in the Action Menu: *Show Osculating Circles with Normals*. In the Animate Menu try the default *Morph*. For related curves see: Parabola, Hyperbola, Conic Sections and their ATOs.

The Ellipse is shown together with the so called *Leitkreis construction* of the curve and its tangent, see below. This construction assumes that the constants aa and bb are positive. The larger of the two is called the semi-major axis length, the smaller one is the semi-minor axis length.
The Ellipse is also the set of points satisfying the following implicit equation: $(x/aa)^2 + (y/bb)^2 = 1$.

A geometric definition of the Ellipse, that can be used to shape flower beds is:

> An Ellipse is the set of points for which the **sum of the distances** from two focal points is a constant L equal to twice the semi-major axis length.

A gardener connects the two focal points by a cord of length L, pulls the cord tight with a stick which then draws the boundary of the flower bed with the stick. Another version of this definition is:

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An Ellipse is the set of points which have **equal distance** from a circle of radius L and a (focal) point inside the circle.

Both these definitions are illustrated in the program.

The normal to an ellipse at any point bisects the angle made by the two lines joining that point to the foci. This says that rays coming out of one focal point are reflected off the ellipse towards the other focal point. Therefore one can build elliptically shaped "whispering galleries", where a word spoken softly at one focal point can be heard only close to the other focal point.

To add a simple proof we show that the tangent leaves the ellipse on one side; more precisely, we show that for every other point on the tangent the sum of the distances to the two focal points F_1, F_2 is more than the length L of the major axis. (In the display: $F = F_2$.) Pick any point Q on the tangent, join it to the two focal points and reflect the segment QF in the tangent, giving another segment QS. Now F_1QS is a radial straight segment only if Q is the point of tangency—otherwise F_1QS is by the triangle inequality longer than the radius F_1S (of length L) of the circle around F_1.

The evolute of an ellipse, i.e., the curve enveloped by the normals of the ellipse—see Action Menu: *Draw osculating circles with normals*, is a generalized Astroid, it is less symmetric than the true Astroid.

An Ellipse can also be obtained by a rolling construction:

Inside a circle of radius aa another circle of radius $r := hh = 0.5aa$ rolls and traces the Ellipse with a stick of radius $R := ii \cdot r$, see Plane Curves Menu: *Circle* and select from the Action Menu: *Show Generalized Cycloids*. The parametric equation resulting from this construction is:

$$x(t) = (R + r)\cos(t)$$
$$y(t) = (R - r)\sin(t)$$

This is related to the visualization of the complex map $z \to z + 1/z$ in Polar Coordinates, the image of the circle of radius R is such an ellipse with $r = 1/R$.

Such rolling constructions are reached with the Plane Curves Menu entry: *Circle* and then the Action Menu *Draw Generalized Cycloids* or with *Epi- and Hypocycloids*. Recall that negative values of the rolling radius hh gives curves on the outside, positive radii ($hh < aa$) on the inside of the fixed circle.
Other rolling curves are:

Cycloid, Astroid, Deltoid, Cardioid, Limacon,
Nephroid, Epi- and Hypocycloids.

H.K.

Parabola*

See also: Ellipse, Hyperbola, Conic Section and their ATOs, and in the Category Surfaces see: Conic Sections and Dandelin Spheres

The usual parametric equations for the Parabola are
$x(t) := t^2/4p$
$y(t) := t,$
where $p = aa/4$,
so the Parabola visualizes the graphs of the two functions $y(x) := \sqrt{4p \cdot x}$ and $x(y) := y^2/4p$.
The vertical line $x = -p$ is called the directrix and the point $(x, y) = (p, 0)$ is called focal point of the Parabola. The distance from a point $(x, y = \sqrt{4p \cdot x})$ on the Parabola to the directrix is $(x + p)$, and this is the same distance as from $(x, y = \sqrt{4p \cdot x})$ to the focal point $(p, 0)$, because $(x - p)^2 + y^2 = (x + p)^2$.
The point $(p, 0)$ is called "focal point", because light rays which come in parallel to the x-axis are reflected off the Parabola so that they continue to the focal point. This fact is illustrated in the program. It gives the following ruler construction of the Parabola:
Prepare the construction by drawing x-axis, y-axis, directrix and focal point F. Then draw any line parallel to the x-axis and intersect it with the directrix in a point S. The

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line orthogonal to the connection SF and through its midpoint is the tangent of the Parabola and intersects therefore the incoming ray in the point of the Parabola which we wanted to find.

The same construction works for Ellipse and Hyperbola, if the directrix is replaced by a circle of radius 2*a around one focal point. The curve is the set of points which have the same distance from this circle and the other focal point.

The Action Menu of the Parabola has an entry "Show Normals Through Mouse Point". This illustrates an unexpected property of the Parabola. One may already be surprised that at the intersection points of normals always **three** normals meet. We know no other curve which is accompanied by such a net of normals. The surprise should increase if one looks at the y-coordinates of the parabola points from where three such intersecting normals originate: these y-coordinates add up to 0! In other words, the intersection behaviour of the normals reflects the addition on the y-axis.

The explanation of where this intersection property comes from is quite interesting. The normals of the Parabola are the tangents to its evolute, the semi-cubical parabola, a singular cubic curve (see Cuspidal Cubic). So the intersection property of the parabola normals can be thought of as defining an addition law for the evolute, and as such it is a simpler limiting case of the addition law that exists on any cubic curve.

Hyperbola*

See also Parabola, Ellipse, Conic Sections and their ATOs.

The most common parametric equations for a Hyperbola with semi-axes aa and bb are:

$x(t) = \pm aa \cosh(t), \quad y(t) = bb \sinh(t), \qquad\qquad t \in \mathbb{R};$

and another version is:

$x(t) = aa/\cos(t),\ y(t) = bb\sin(t)/\cos(t), \quad t \in [0, 2\pi].$

The corresponding implicit equation is:

$(x/aa)^2 - (y/bb)^2 = 1.$

The function graphs: $\{(x, y);\ y = 1/x + m \cdot x\}$ are also Hyperbolae.

A geometric definition of the Hyperbola is:

> A Hyperbola is the set of points for which the **difference of the distance** from two focal points is constant.

Or:

> A Hyperbola is the set of points which have the **same distance** from a circle and a (focal) point **outside** that circle.

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If one applies an inversion $(x, y) \to (x, y)/(x^2 + y^2)$ to a right Hyperbola (i.e. $aa = bb$) then one obtains a Lemniscate.

In the visualization of the complex map $z \to z + 1/z$ in Polar Coordinates, the image of the radial lines are the Hyperbolae:

$$x(R) = (R + 1/R)\cos\phi$$
$$y(R) = (R - 1/R)\sin\phi, \qquad R \in \mathbf{R}.$$

And the image of the standard Cartesian Grid under the complex map $z \to \sqrt{z}$ is a grid of two families of orthogonal Hyperbolae.

(H.K.)

Conic Sections, 2D construction*

See also Parabola, Ellipse, Hyperbola and their ATOs.

A cone of revolution (e.g., $\{(x, y, z); x^2+y^2 = m\cdot z^2\}$) is one of the simplest surfaces. Its intersections with planes are called conic sections. Apart from pairs of lines these conic sections are Parabolae, Ellipses or Hyperbolae. These curves have also other geometric definitions (e.g., The locus of points having the same distance from a focal point and a circle). See their Menu entries.

On the other hand, they are also more robust than these definitions show: Photographic images of conic sections are again conic sections; or in a completely different formulation: The intersection of a plane and any "quadratic cone", i.e.,
$\{(x, y, z) \mid a \cdot x^2 + b \cdot y^2 + c \cdot z^2 + d \cdot xy + e \cdot yz = 0\}$,
is **not** more complicated than planar sections of circular cones but are the same old Parabolae, Ellipses or Hyperbolae as above. A special case of this robustness is the fact that orthogonal projections of conic sections in 3-space are again conic sections. This is illustrated in the program as

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follows:

Interpret the illustration as if it showed level lines on a hiking map. The equidist parallel lines are the level lines of a sloping plane; the smaller the distance between these level lines the steeper the plane. The equidistant concentric circles are the level lines of a circular cone, as for example an ant lion would dig in sandy ground; without height numbers written next to the level lines we can of course not decide whether the circular level lines represent a conical mountain or a conical hole in the ground. We suggest that the blue level line and the vertex of the cone are at height zero and the other levels are higher up so that the cone is a hole.

The intersection curve between plane and cone has then an easy pointwise construction: Simply intersect level lines of the same height on the two surfaces. (These are lines with the same color in the program illustration.) This construction reveals a new geometric property of the intersection curve on the map, of this conic section:

> Take the ratio of the distances from a point on the curve, (i) to the level line at height 0 of the plane (called *directrix*) and (ii) to the vertex at height zero of the cone (called *focus*). *This ratio is the same* as the ratio of adjacent level lines of plane and cone and therefore the same *for all points of this conic section.*

Conic Sections, Kepler orbits*

See also Parabola, Ellipse, Hyperbola and their ATOs.

For many properties of the conic sections a parametrization is not relevant. However, when Kepler discovered that planets and comets travel on conic sections around the sun then this discovery came with a companion: the speed on the orbit is such that angular momentum is preserved. In more elementary terms: the radial connection from the sun to the planet sweeps out equal areas in equal times. With the 3dfs demo we explain geometrically how this celestial parametrization is connected with the focal properties of conic sections. Here we give the *algebraic explanation* first.

An ellipse, parametrized as *affine image of a circle* and translated to the left is

$$P(\varphi) := (a\cos\varphi - e,\ b\sin\varphi).$$

If we choose $e := \sqrt{a^2 - b^2}$ then we have $|P(\varphi)| = (a - e\cos\varphi)$. This gives the connection with the oldest definition of an ellipse: The sum of the distances from $P(\varphi)$ to the two points $(\pm e, 0)$ is $2a$.

Next we compute the quantity A, equal to twice the area swept out by the position vector P, and also the derivatives

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of P and A:

$$A(\varphi) = \int_0^\varphi \det(P(\varphi), P'(\varphi))d\varphi,$$
$$P'(\varphi) = (-a\sin\varphi, b\cos\varphi),$$
$$A'(\varphi) = b(a - e\cos\varphi),$$

and we denote the function inverse to $A(\varphi)$ by $\Phi(A)$, so that,

$$\Phi(A(\varphi)) = \varphi, \qquad \Phi'(A) = \frac{1}{b(a - e\cos\Phi)}.$$

Let us write Q to denote the position when expressed as a function of A, i.e., $Q(A) := P(\Phi(A))$. Now Kepler's Second Law says that A proportional to time, or equivalently that A *is* the time in approriate units, so the velocity is $Q'(A) = P'(\Phi(A)) \cdot \Phi'(A)$, and the kinetic energy is:

$$\begin{aligned} K.E. = \frac{1}{2}Q'(A)^2 &= \frac{a^2\sin^2\varphi + b^2\sin^2\varphi}{2b^2(a - e\cos\varphi)^2} \\ &= \frac{a^2 - e^2\cos^2\varphi}{2b^2(a - e\cos\varphi)^2} \\ &= \frac{(a + e\cos\varphi)}{2b^2(a - e\cos\varphi)} \\ &= \frac{a}{b^2} \cdot \frac{1}{a - e\cos\varphi} - \frac{1}{2b^2}. \\ &= \frac{a}{b^2} \cdot \frac{1}{|P(\varphi)|} - \frac{1}{2b^2}. \end{aligned}$$

Thus, in units where we take twice the swept out area as the time, the potential energy can be read off by using the law of energy consevation, i.e., the fact that the kinetic energy plus the potential energy is constant. In fact, it follows from this that the potential energy at orbit point $Q(A(\varphi)) = P(\varphi)$ is equal to:

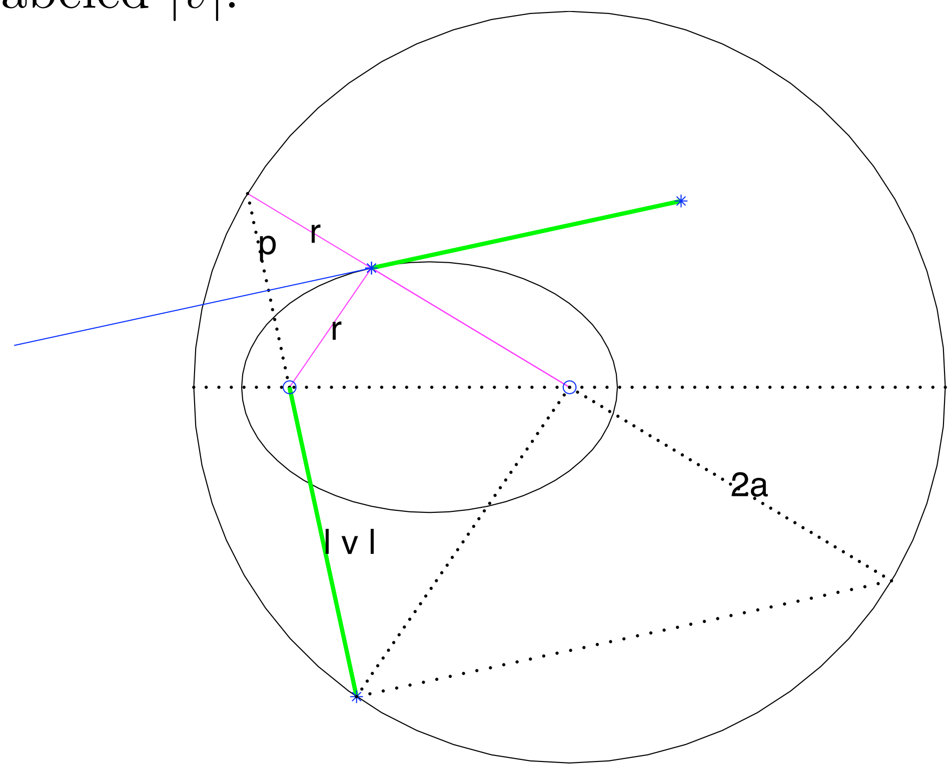
$$-\frac{a}{b^2} \cdot \frac{1}{|P(\varphi)|},$$

which is the famous $1/r$ law for the potential energy.

Next, we present a *geometric proof*. The starting point is the determination of the correct orbital speed by the property that the product of the speed $|v|$ with the distance p of the tangent line from the center is the constant angular momentum, Kepler's second law. Of course we can illustrate such a fact only if we also represent the size of velocities by the length of segments and we have to keep in mind that segments which illustrate a length and segments which illustrate a velocity are interpretated with different units.

Recall the following theorem about circles: if two secants of a circle intersect then the product of the subsegments of one secant ist the same as the product of the subsegments of the other secant.

This will be applied to the circle the radius of which is the length $2a$ of the major axis. (The midpoint is the other focus, not the sun.) The two secants intersect in the focus representing the sun: one secant is an extension of the major axis the other is perpendicular to the tangent line. The subsegments of the first secant have the lengths $2a-2e$ and $2a+2e$, where $2e$ is the distance between the foci. The subsegments of the second secant have one length $2p$ and one labeled $|v|$.



Kepler Ellipse with construction
of proper speed and potential.

The circle theorem says: $(2a - 2e) \cdot (2a + 2e) = 2p \cdot |v|$. Since the left side is constant we can interprete the segment labeled $|v|$ as representing the *correct orbital speed.*

Now that we know at each point of the orbit the correct speed we can deduce Newton's $1/r$-law for the gravitational potential, if we use *kinetic energy plus potential energy equals constant total energ.* In the illustration we have two similar right triangles, the small one has hypothenuse $= r$ and one other side $= p$, the big one has as hypothenuse a circle diameter of length $4a$ and the corresponding other side has length $2p + |v|$. Now we use the above $const := (2a - 2e) \cdot (2a + 2e) = 2p \cdot |v|$ to eliminate p from the proportion:

$$p : r = (2p + |v|) : 4a$$

This gives

$$2a/r = 1 + |v|/2p = 1 + v^2/const.$$

Up to physical constants (units), v^2 is the kinetic energy, so that (again up to units) $-1/r$ is the potential energy – since such a potential makes kinetic plus potential energy constant.

Another simple property of Kepler ellipses and hyperbolas is: Their velocity diagram, the so called *hodograph*, is a circle. Usually one simply translates the velocity vector from the orbit point to the sun. In our picture we see the velocity vector rotated by 90 degrees; indeed, it ends on the circle. This leads to a geometric representation of the *Runge-Lenz vector*: In our picture we really see the cross product of the (tangential) velocity vector with angular momentum (a constant vector orthogonal to the orbit plane). If we add to it a vector of constant length $2a$ and *parallel* to the position vector then we reach the midpoint of our circle, the other focal point of the orbit ellipse. This sum vector is, up to the constant negative factor $-(a - e)/2e$ the classical Runge-Lenz vector.

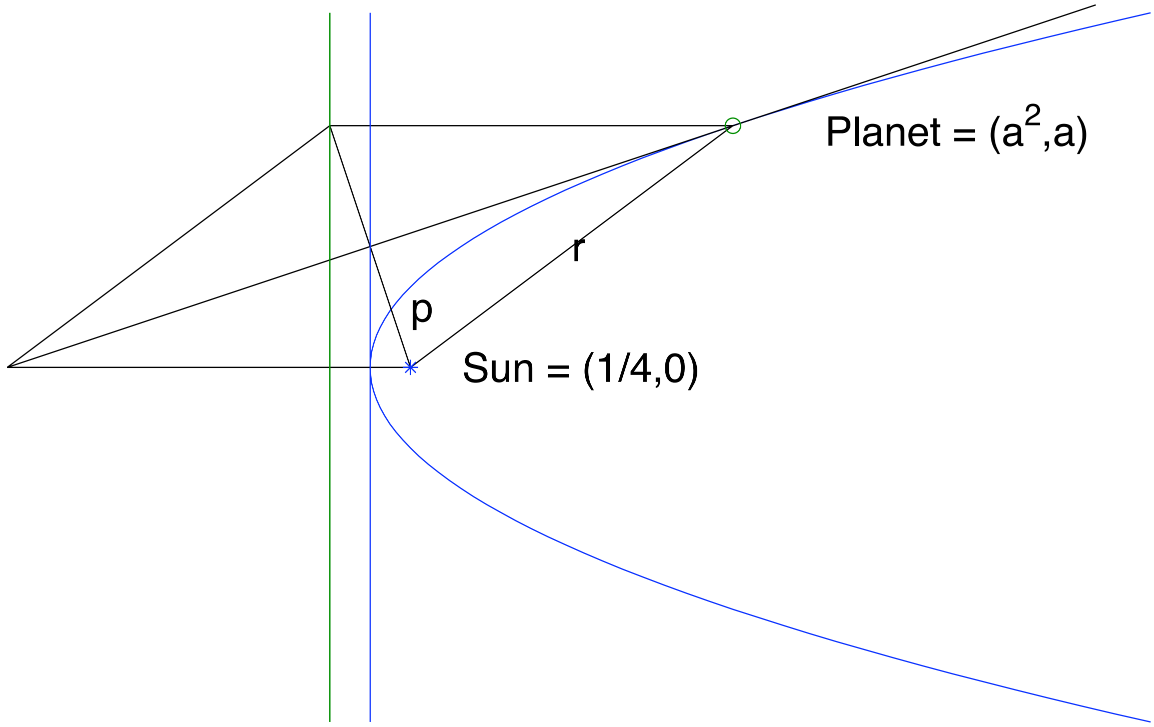
Mathematically, the *parabolic* and *hyperbolic* Kepler orbits allow similar derivations of the $-1/r$-potential, which we will give next. Historically this played no role since the non-repeating orbits could not be determined with enough precision at the time.

Derivation of the $-1/r$-potential from a parabolic Kepler orbit. Let in the picture (below) $|p|$ be the distance from the sun at $(1/4, 0)$ to a tangent of a parabolic Kepler orbit and let $|v|$ be the orbital speed at that moment. Conservation of angular momentum says $p \cdot |v| = const.$ Let φ be the angle between the segment marked p and the vertical axis; since the sun is at the focal point of the parabola we have $p \cdot \sin\varphi = 1/4$. This and the previous angular momentum equation say that, up to a choice of unit for velocity, we have:

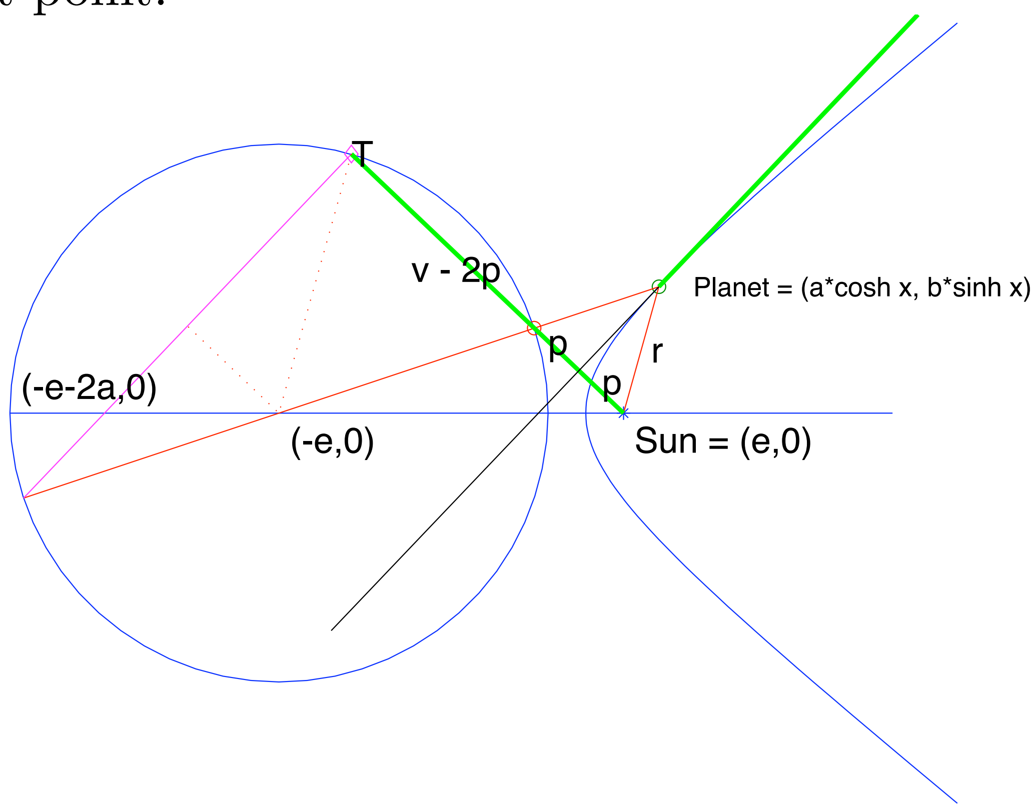
Kepler speed: $|v| = \sin\varphi$,
Angular momentum: $p \cdot |v| = 1/4$.

If we call r the distance to the planet, than we also have $p/r = \sin\varphi = |v|$. Multiplication with the angular momentum gives

Kinetic energy: $\frac{1}{2}|v|^2 = \frac{1}{8r}$,
Potential energy: $\frac{-1}{8r}$.



Derivation of the $-1/r$-potential from a hyperbolic Kepler orbit. As before we call p the distance from the sun to a tangent of the hyperbolic orbit and v the speed at that orbit point.



Conservation of angular momentum says $p \cdot |v| = const.$ We use the property of the circle (radius $2a$) about products of segments on secants (which intersect at the sun S):

$2p \cdot |T - S| = (2e - 2a)(2e + 2a) = 4b^2$. Therefore, again up to the unit for velocity, we have identified the correct

$$\textbf{Kepler velocity: } v = |S - T|.$$

Finally, similar triangles give:

$p/r = (v - 2p)/4a \quad \text{or} \quad 4a/r = v/p - 2,$

and elimination of p with the angular momentum, i.e. with $1/p = v/2b^2$, shows that kinetic energy plus a radial function are constant – thus identifying the $1/r$-potential:

$$4ab^2/r = v^2/2 - 2b^2.$$

Additional properties: As in the case of elliptical orbits we see that the *hodograph* is a circle because the velocity vector, rotated by 90 degrees, ends on the circle which we used for the construction of the hyperbola. And if we add to the endpoint of this rotated velocity a vector parallel to the position vector and of constant length 2a then we reach the midpoint of the circle, the other focal point of the orbit. The constant(!) difference vector between the two focal points, the geometric *Runge-Lenz vector*, differs from the common definition by the constant factor $(e - a)/2e$.

Nephroid of Freeth*

This curve, first described 1879, is the member $aa = 0$ in the following family of curves:

$$x(t) = (1 - aa \cdot \sin(t/2)) \cos(t)$$
$$y(t) = (1 - aa \cdot \sin(t/2)) \sin(t)$$

The default morph starts at $aa = 0$ with a circle, traversed twice. For small $aa > 0$ one double point develops. At $aa = 1$ the curve reaches the origin with a cusp. This cusp deforms into a second double point. At $aa = \sqrt{2}$ the two tangents of the double point coincide and are vertical. This point of double tangency deforms into three double points. The Nephroid of Freeth is reached at $aa = 2$, when two of the mentioned three double points coincide with the earliest one to form a *triple intersection.*
Apart from being in a simple family, which shows all these singularities of curves, we learnt from

www.2dcurves.com/derived/strophoid.html

that the Nephroid of Freeth has the curious property that one can construct a regular sevengon with it: The vertical tangent at the triple intersection meets the curve again in two points whose radius vectors enclose the angle $3\pi/7$.

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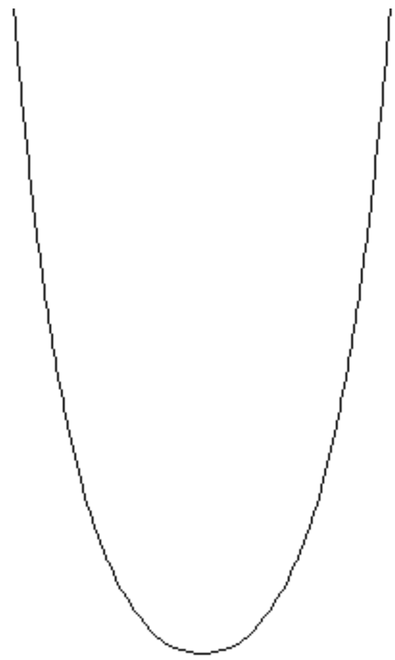
Sine Curve text is not completed.

HK

Catenary*

The catenary is also known as the chainette, alysoid, and hyperbolic cosine. It is defined as the graph of the function $y = a\cosh(x/a)$. (Recall $\cosh(x) := (e^x + e^{-x})/2$, where $e = 2.71828\ldots$ is the base of the natural logarithms.)



The Catenary

The catenary is the shape an ideal string takes when hanging between two points. By "ideal" is meant that the string is perferctly flexible and inextensible, has no thickness, is of uniform density. In other words the catenary is a mathematical abstraction of the shape of a hanging string, and it closely approximates the shapes of most hanging string-like

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

objects we see, such as ropes, outdoor telecommunication wires, necklaces, chains, etc. For any particular hanging string, we will need to choose the parameter a correctly to model that string.

Notice that except for scaling there is really only a single catenary. That is, the scaling transformation $(x, y) \mapsto (ax, ay)$ maps the graph of $y = a\cosh(x/a)$ onto the graph of $y = \cosh(x)$. The scaling transformation just amounts to a change in the choice of units used to measure distances.

History

Galileo was the first to investigate the catenary, but he mistook it for a parabola. James Bernoulli in 1691 obtained its true form and gave some of its properties. [cf., Robert C. Yates, 1952]

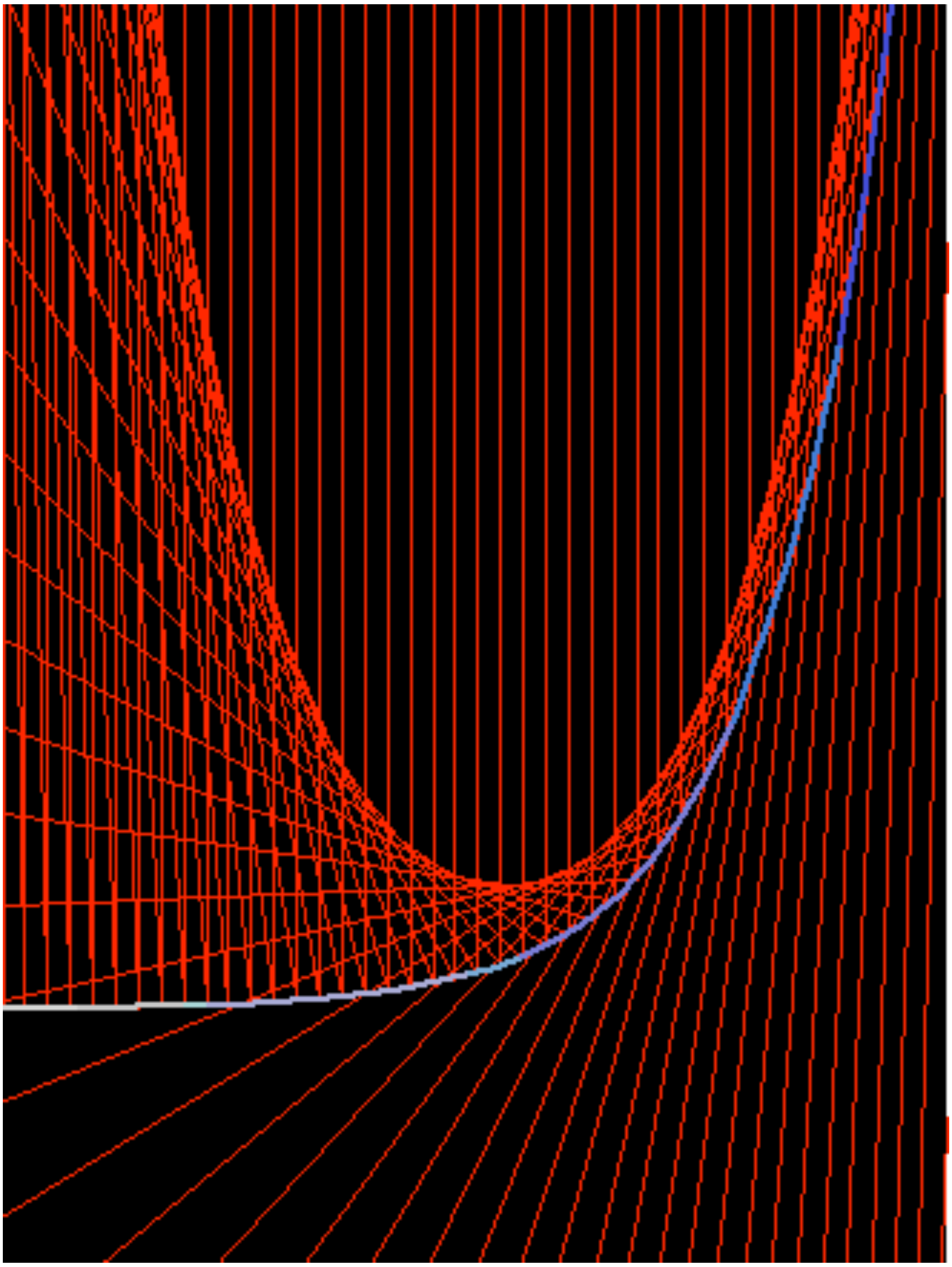
Galileo's suggestion that a heavy rope would hang in the shape of a parabola was disproved by Jungius in 1669, but the true shape of the catenary, was not found until 1690–91, when Huygens, Leibniz and John Bernoulli replied to a challenge by James Bernoulli. David Gregory, the Oxford professor, wrote a comprehensive treatise on the 'catenarian' in 1697. The name was first used by Huygens in a letter to Leibniz in 1690. [cf., E.H.Lockwood, 1961].

[By the way, it is true that if you carefully weight a hanging string so that their is equal weight of string per unit of horizontal distance (rather than per unit of length) then its shape will be a parabola, so Gallileo wasn't so far from the truth.]

The Catenary has numerous interesting properties.

Properties of the Catenary

Caustics

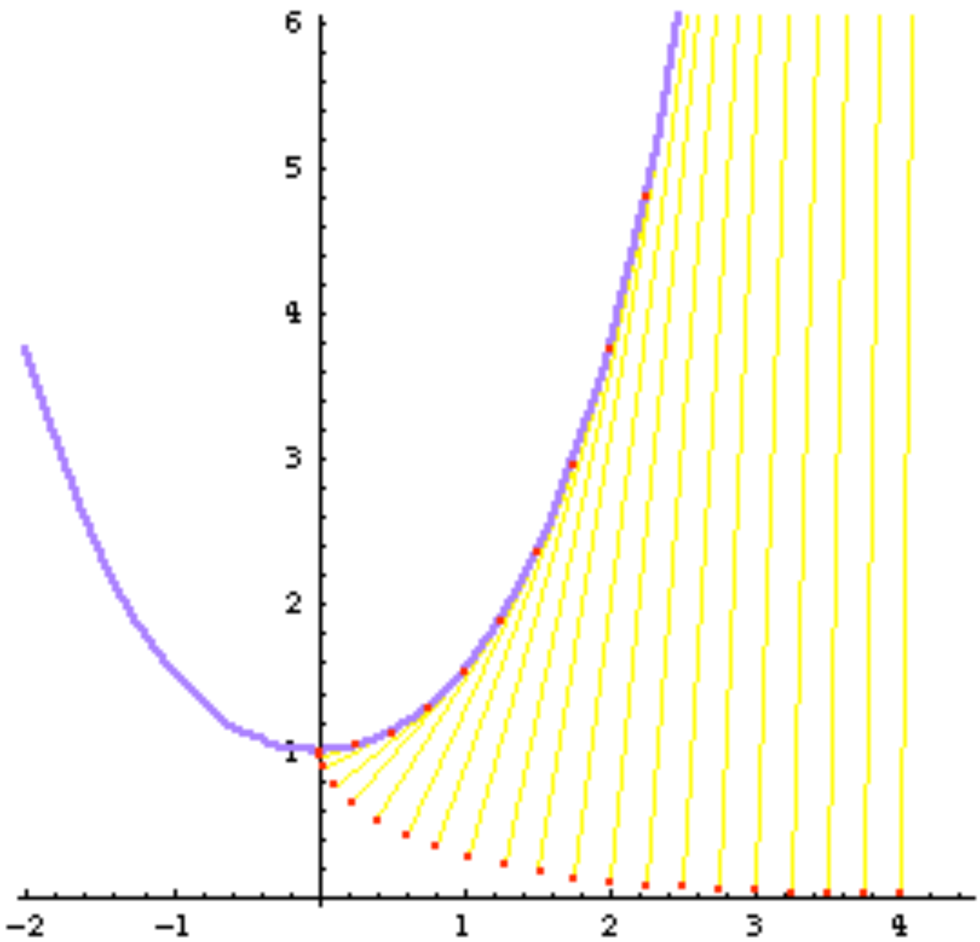


Parallel rays above the exponential curve

The Catacaustic of the exponential curve (x, e^x) with light rays from above and parallel to the y axes is the catenary.

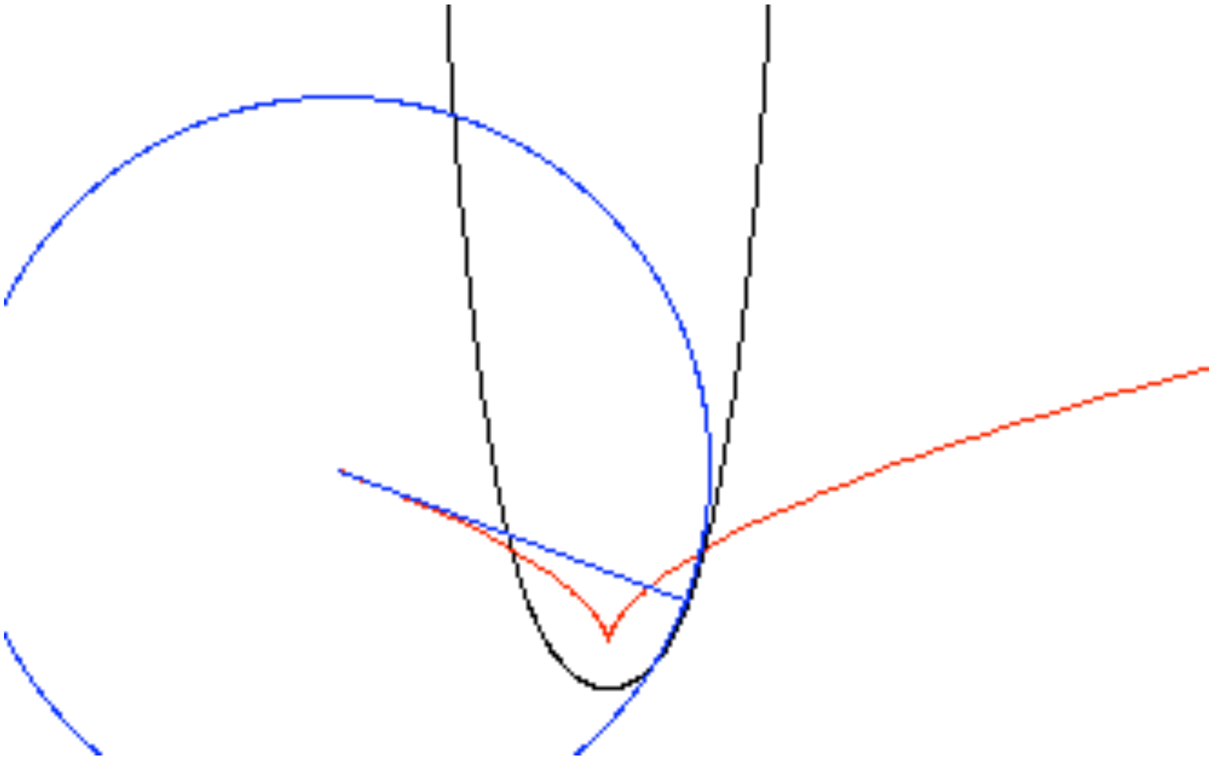
The exponential function e^x has interesting properties itself. It is the only function who agrees with its derivative.

Involute



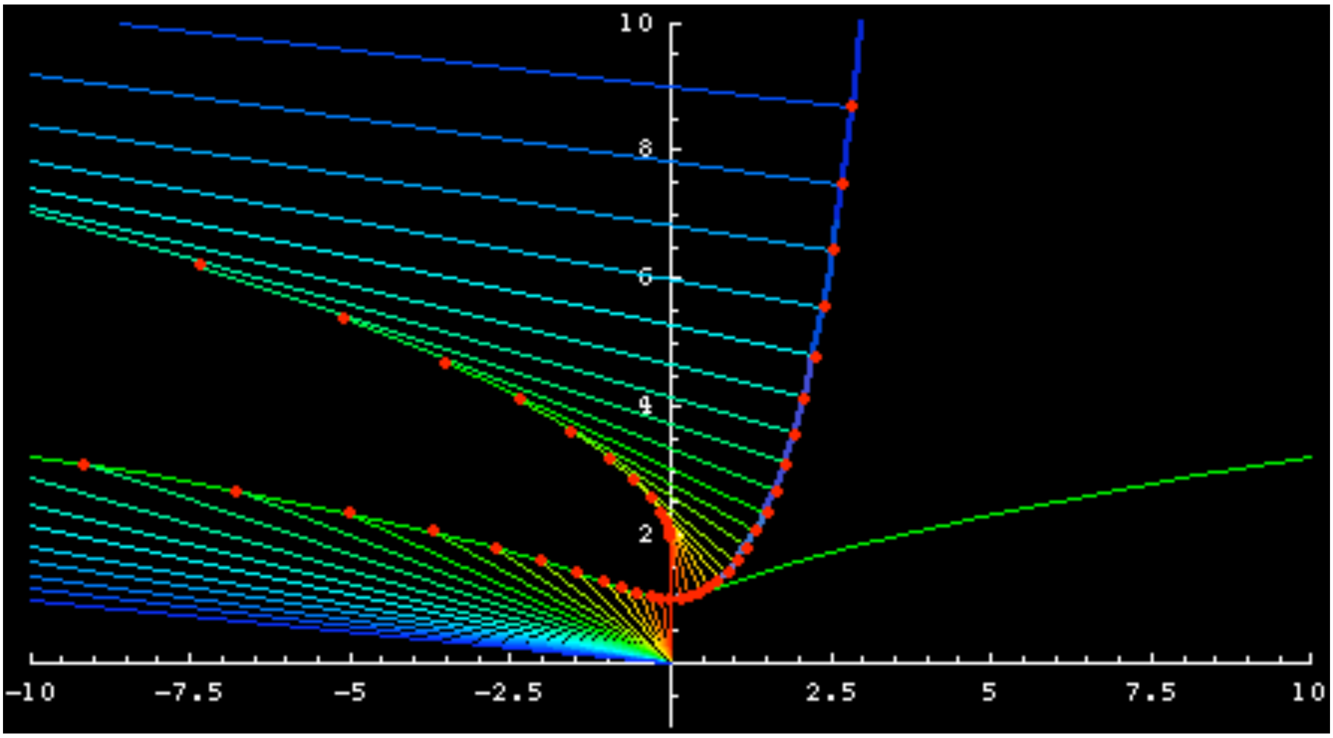
The involute of catenary starting at the vertex is the curve tractrix. (In 3DXM, the involutes of a curve can be shown in the menu Action → Show Involutes.) Note that all involutes are parallel curves of each other. This is a theorem.

Evolute



The evolute of the catenary is also the tractrix. (In 3DXM, this can be seen from the menu Action → Show Osculating Circles.)

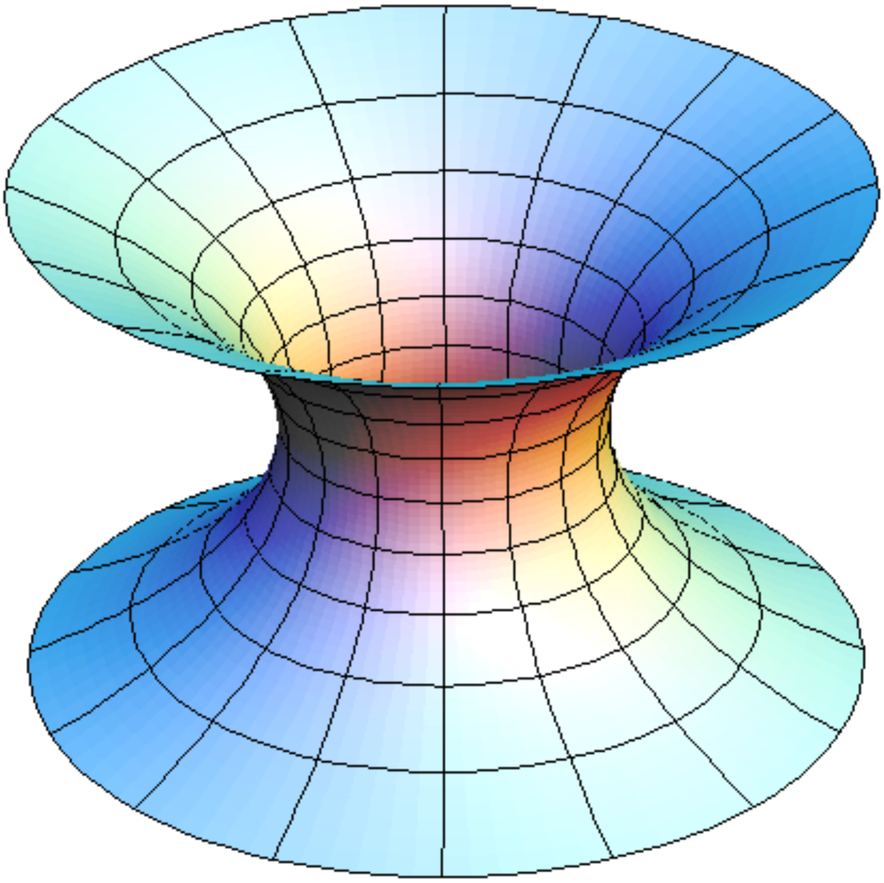
Radial and Kampyle of Eudoxus



The radial of the catenary is the Kampyle of Eudoxus. In the figure above, the blue curve is half the catenary. The green curve is the Kampyle of Eudoxus. The rainbow lines are radii of osculating circles and their parallels through 0.

The Kampyle of Eudoxus is defined as the parametric curve $x = -\cosh(t)\sinh(t), y = \cosh(t)$.

Catenoid



If you rotate the graph of $x = \cosh(y)$ about the y-axis, the resulting surface of revolution is a minimal surface, called the Catenoid. It is one endpoint of an interesting morph you can see in 3DXM, by switching to the Surface category, choosing Helicoid-Catenoid from the Surface menu, and then choosing Morph from the Animate menu. If you look closely you will see that during this morph distances and angles on the surface are preserved. See About This Object... in the Documentation menu when Helicoid-Catenoid is selected for a discussion of this.

XL.

On Curves Given By Their Support Function*

This note is about smooth, closed, convex curves in the plane and how to define them in terms of their so-called *Minkowski support function h*. For quick reference we first show how, in 3D-XplorMath, h can be modified by specifying parameters. Then we begin with a more general class of geometric objects, namely *convex bodies*.

1. Parameter Dependent Formulas

In 3D-XplorMath, the support function h is given in terms of Fourier summands:

$$\begin{aligned} h(\varphi) := \ & aa + bb\cos(\varphi) + cc\cos(2\varphi) + \\ & dd\cos(3\varphi) + ee\cos(4\varphi) + ff\cos(5\varphi). \end{aligned}$$

In terms of this function we define the following curve:

$$c(\varphi) := h(\varphi) \cdot \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix} + h'(\varphi) \cdot \begin{pmatrix} -\sin(\varphi) \\ +\cos(\varphi) \end{pmatrix}.$$

Differentiation shows that c is given in terms of its unit normal and tangent vectors and the function h:

$$c'(\varphi) = (h + h'')(\varphi) \cdot \begin{pmatrix} -\sin(\varphi) \\ +\cos(\varphi) \end{pmatrix}.$$

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

One obtains curves with nonsingular parametrization ($|c'| > 0$) if aa is choosen large enough. And since $h(\varphi)$ equals the scalar product between $c(\varphi)$ and the unit normal $n(\varphi) = (\cos(\varphi), \sin(\varphi))$ one has a simple geometric interpretation: $h(\varphi)$ is the distance of the tangent at $c(\varphi)$ from the origin.

2. Background And Explanations

A convex body in $\mathbb{R}^n$ is a compact subset $\mathbb{B}$ having non-empty interior and such that it includes the line segment joining any two of its points. A hyperplane H in $\mathbb{R}^n$ is called a supporting hyperplane of $\mathbb{B}$ if it contains a point of $\mathbb{B}$ and if $\mathbb{B}$ is included in one of the two halfspaces defined by H. It is not difficult to show that every boundary point of $\mathbb{B}$ lies on at least one supporting hyperplane, and that $\mathbb{B}$ is the intersection of all such halfspaces.

A smooth, closed, planar curves c is called *convex* if its tangent at each point intersects c only at that one point. The complement in $\mathbb{R}^2$ of such a curve has a single bounded component, the *interior* of the curve, and one unbounded component, its *exterior*. The curve is the boundary of its interior, and we denote by $\mathbb{B}$ the curve together with its interior. It is easy to see that $\mathbb{B}$ is a convex body in $\mathbb{R}^2$, as defined above, and in fact the tangent line at any point of c is the unique supporting hyperplane (= line!) containing that point. (There are of course more general planar convex bodies. For example if P is a closed polygon in $\mathbb{R}^2$ together with its interior, then P is a convex body, but there are infinitely many supporting lines through each vertex, while the supporting line containing an edge contains infinitely many points.)

Now let O be some interior point of c and take O as the origin of a cartesian coordinates by fixing a ray from O as the positive x-axis. With respect to these coordinates, at each point p on c the outward directed unit normal at p will have the form $n(\varphi) = (\cos(\varphi), \ \sin(\varphi))$ where $\varphi = \varphi(p)$ satisfies $0 \leq \phi \leq 2\pi$. If we as usual think of $\mathbb{S}^1$ as the interval $[0, 2\pi]$ with endpoints identified, then it can be shown that the map $p \mapsto \varphi(p)$ is a smooth one-to-one map of c with $\mathbb{S}^1$, so that the inverse map gives a parametrization $c(\varphi)$ of the curve by $\mathbb{S}^1$. (This just says that given any direction in the plane, there is a unique point p on c where the outward normal has that direction, and the point p varies smoothly with the direction.)

The Minkowski *support function* for the curve c is the function h defined on $\mathbb{S}^1$ by letting $h(\varphi)$ be the distance from the origin of the line of support (or tangent) through $c(\varphi)$, that is $h(\varphi) := n(\varphi) \cdot c(\varphi)$, the scalar product of $c(\varphi)$ and $n(\varphi)$. From this definition it is easy to reconstruct the curve in terms of its support function as in part 1.

3. Things To Observe

Recall one has in any parametrization the curvature formula

$$n'(t) = \kappa(t) c'(t),$$

which in the present case reduces to:

$$1/\kappa(\varphi) = h(\varphi) + h''(\varphi) = |c'(\varphi)|.$$

Clearly aa has to be large enough to make κ positive and the parametrization nonsingular. Adding a linear combination of $\cos(\varphi)$ and $\sin(\varphi)$ to the support function corresponds to a change of only the origin, the shape of the curve stays the same. The $bb\cos(\varphi)$-term in the support function is therefore not really necessary, but one can use it to see how the parametrization of the curve changes.

The cos-terms in even multiples of φ make up the even part $(h(\varphi)+h(\varphi+\pi))/2$ of h. The origin is the *midpoint* of curves with even support function. If h is odd except for the constant term, i.e.,

$$h(\varphi) = aa + (h(\varphi) - h(\varphi+\pi))/2,$$

then one obtains curves of constant width w where:

$$w = h(\varphi) + h(\varphi+\pi) = 2\,aa.$$

The default curve in 3D-XplorMath is such a curve of constant width and the default morph shows a family of such curves. We emphasize the width of our curves by drawing them together with their pairs of parallel tangents. Since the (non-)constancy of the distance between these parallel tangents is difficult to see we have added a circle of the same width (= diameter). One cannot easily recognize how many extrema the curvature $\kappa(\varphi)$ has. To see it clearly we recommend selecting the entry *Show Osculating Circles* from the Action Menu, since the evolute has a cusp at every extremal value of κ.

Tractrix*

The Tractrix is a curve with the following nice interpretation: Suppose a dog-owner takes his pet along as he goes for a walk "down" the y-axis. He starts from the origin, with his dog initially standing on the x-axis at a distance aa away from the owner. Then the Tractrix is the path followed by the dog if he "follows his owner unwillingly", i.e., if he constantly pulls against the leash, keeping it tight. This means mathematically that the leash is always tangent to the path of the dog, so that the length of the tangent segment from the Tractrix to the y-axis has constant length aa. Parametric equations for the Tractrix (take $bb = 0$) are:

$$x(t) = aa \cdot \sin(t)(1 + bb)$$
$$y(t) = aa \cdot (\cos(t)(1 + bb) + \ln(\tan(t/2))).$$

The curves obtained for $bb \neq 0$ are generated by the same kinematic motion, except that a different point of the moving plane is taken as the drawing pen. See the default `Morph`.

The Tractrix has a well-known surface of revolution, called the Pseudosphere, Namely, rotating it around the y-axis gives a surface with Gaussian curvature -1. This means

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

that the Pseudosphere can be considered as a portion of the Hyperbolic Plane. The latter is a geometry that was discovered in the 19th century by Bolyai and Lobachevsky. It satisfies all the axioms of Euclidean Geometry except the Axiom of Parallels. In fact, through a point outside a given line (= geodesic) there are infinitely many lines that are parallel to (i.e., do not meet) the given line.

There are many connections, sometimes unexpected, between planar curves. For the Tractrix select: `Show Osculating Circles And Normals`. One observes a Catenary (see another entry in the curve menu) as the envelope of the normals.

H.K.

Cissoid and Strophoid*

3DXM Family: $c(t) := 2aa\left(\dfrac{t(t^2-bb)}{(1+t^2)},\ \dfrac{bb}{2aa}+\dfrac{t^2-bb}{(1+t^2)}\right)$

The additive constant $bb/2aa$ in the y-coordinate has the effect that the drawing mechanism is the same for the whole family, try the default `Morph` in the Animate Menu.

History

Diocles (250 – ∼100 BC) invented the Cissoid to solve the doubling of the cube problem (also know as the the Delian problem). The name Cissoid (ivy-shaped) derives from the shape of the curve. Later the method used to generate this curve was generalized, and we call all curves generated in a similar way Cissoids. Newton (see below) found a way to generate the Cissoid mechanically. The same kinematic motion with a different choice of the drawing pen generates the (right) Strophoid, formulas below.

From Thomas L. Heath's Euclid's Elements translation (1925) (comments on definition 2, book one):

> This curve is assumed to be the same as that by means of which, according to Eutocius, Diocles in his book *On Burning-Glasses* solved the problem of doubling the cube.

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

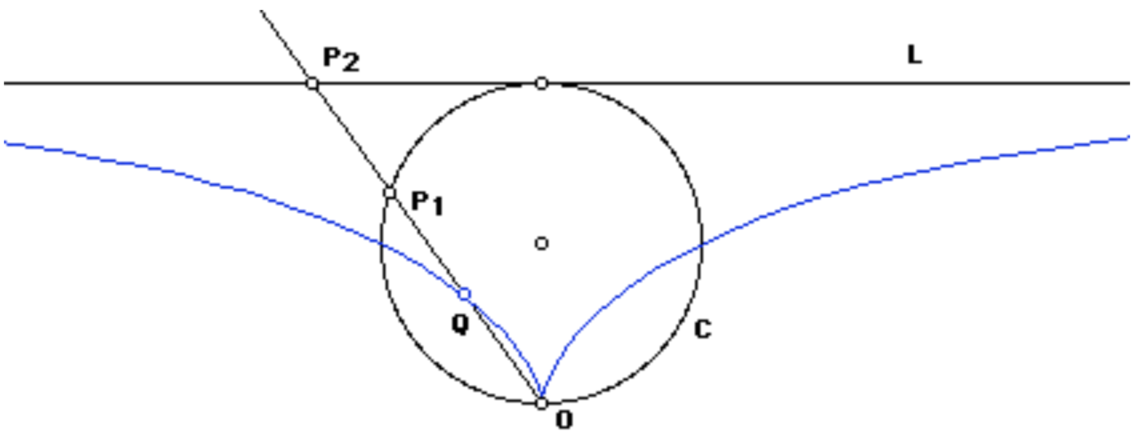
From Robert C. Yates' Curves and their properties (1952):

> As early as 1689, J. C. Sturm, in his *Mathesis Enucleata*, gave a mechanical device for the constructions of the Cissoid of Diocles.

From E.H.Lockwood A book of Curves (1961):

> The name cissoid ("Ivy-shaped") is mentioned by Geminus in the first century B.C., that is, about a century after the death of the inventor Diocles. In the commentaries on the work by Archimedes *On the Sphere and the Cylinder*, the curve is referred to as Diocles' contribution to the classic problem of doubling the cube. ... Fermat and Roberval constructed the tangent (1634); Huygens and Wallis found the area (1658); while Newton gives it as an example, in his *Arithmetica Universalis*, of the ancients' attempts at solving cubic problems and again as a specimen in his Enumeratio Linearum Tertii Ordinis.

1 Description



The Cissoid of Diocles is a special case of the general cissoid. It is a cissoid of a circle and a line tangent to the circle with respect to a point on the circle opposite to the tangent point. Here is a step-by-step description of the construction:

1. Let there be given a circle C and a line L tangent to this circle.
2. Let O be the point on the circle opposite to the tangent point.
3. Let P1 be a point on the circle C.
4. Let P2 be the intersection of line [O,P1] and L.
5. Choose Q on line [O,P1] with $\text{dist}[O, Q] = \text{dist}[P1, P2]$.
6. The locus of Q (as P1 moves on C) is the cissoid of Diocles.

An important property to note is that Q and P1 are symmetric with respect to the midpoint of the segment [O,P2]. Call this midpoint M. We can reflect every element in the construction around M, which will help us visually see other properties.

2 Formula derivation

Let the given circle C be centered at $(1/2, 0)$ with radius $1/2$. Let the given line L be $x = 1$, and let the given point O be the origin. Let $P1$ be a variable point on the circle, and Q the tracing point on line $[O, P1]$. Let the point $(1, 0)$ be A. We want to describe distance $r = \text{dist}[O, Q]$ in terms of the angle $\theta = [A, O, P1]$. This will give us an equation for the Cissoid in polar coordinates (r, θ). From elementary geometry, the triangle $[A, O, P1]$ is a right triangle, so by trignometry, the length of $[O, P1]$ is $\cos(\theta)$. Similarly, triangle $[O, A, P2]$ is a right triangle and the length of $[O, P2]$ is $\frac{1}{\cos(\theta)}$. Since $\text{dist}[O, Q] = \text{dist}[O, P2] - \text{dist}[O, P1]$, we have $\text{dist}[O, Q] = \frac{1}{\cos(\theta)} - \cos(\theta)$. Thus the polar equation is $r = \frac{1}{\cos(\theta)} - \cos(\theta)$. If we combine the fractions and use the identity $\sin^2 + \cos^2 = 1$, we arive at an equivalent form: $r = \sin(\theta)\tan(\theta)$.

3 Formulas for the Cissoid and the Strophoid

In the following, the cusp is at the origin, and the asymptote is $x = 1$. (So the diameter of the circle is 1 ($= aa$ in 3DXM).)

Parametric: $(\sin^2(t), \sin^2(t)\tan(t)) \quad -\pi/2 < t < \pi/2.$

Parametric: $\left(\dfrac{t^2}{(1+t^2)}, \dfrac{t^3}{(1+t^2)}\right) \quad -\infty < t < \infty$

Strophoid: $\left(\dfrac{t^2-1}{(1+t^2)}, \dfrac{t(t^2-1)}{(1+t^2)}\right) \quad -\infty < t < \infty$

Polar: $r = \frac{1}{\cos(\theta)} - \cos(\theta) \quad -\pi/2 < t < \pi/2.$

Cissoid: $y^2(1-x) = x^3$, Strophoid: $y^2(1-x) = x^2(1+x)$.

The Cissoid has numerous interesting properties.

4 Properties

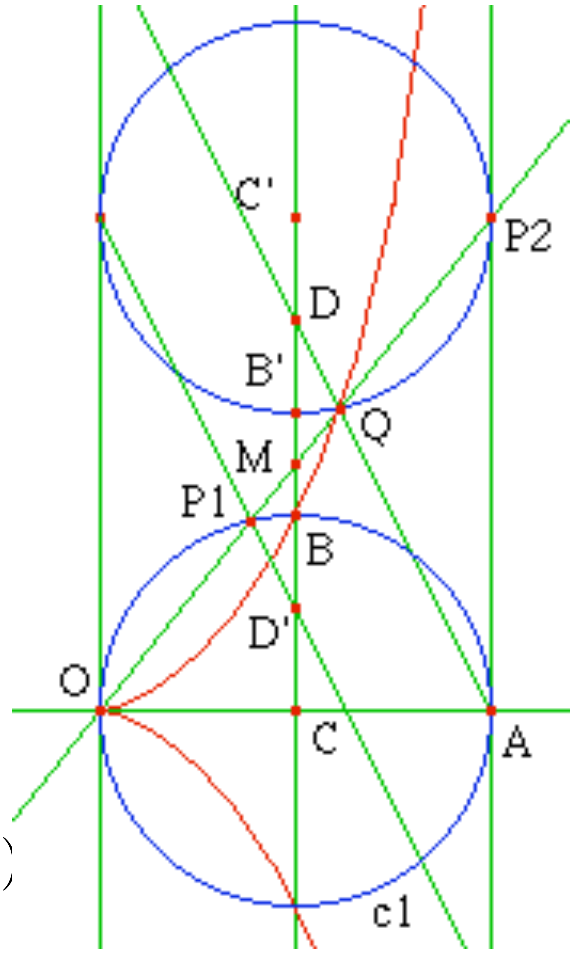
4.1 Doubling the Cube

Given a segment $[C, B]$, with the help of the Cissoid of

Diocles we can construct a segment $[C, M]$ such that $\text{dist}[C, M]^3 = 2 * \text{dist}[C, B]^3$. This solves the famous doubling the cube problem.

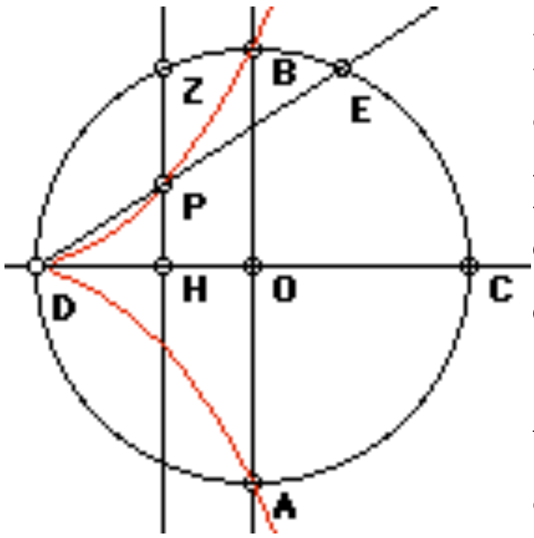
Step-by-step description:

1. Given two points C and B.
2. Construct a circle $c1$, centered on C and passing through B.
3. Construct points O and A on the circle such that line $[O, A]$ is perpendicular to line $[C, B]$
4. Construct a cissoid of Diocles using circle $c1$, tangent at A, and pole at O.
5. Construct point D such that B is the midpoint of segment[C,D].
6. Construct line[A,D]. Let the intersection of cissoid and line[A,D] be Q. (The intersection cannot be found with Greek Ruler and Compass. We assume it is a given.)
7. Let the intersection of line $[C, D]$ and line $[O, Q]$ be M.
8. $\text{dist}[C, M]^3 = 2 \cdot \text{dist}[C, D]^3$.



This can be proved trivially with analytic geometry.

4.2 Diocles' Construction



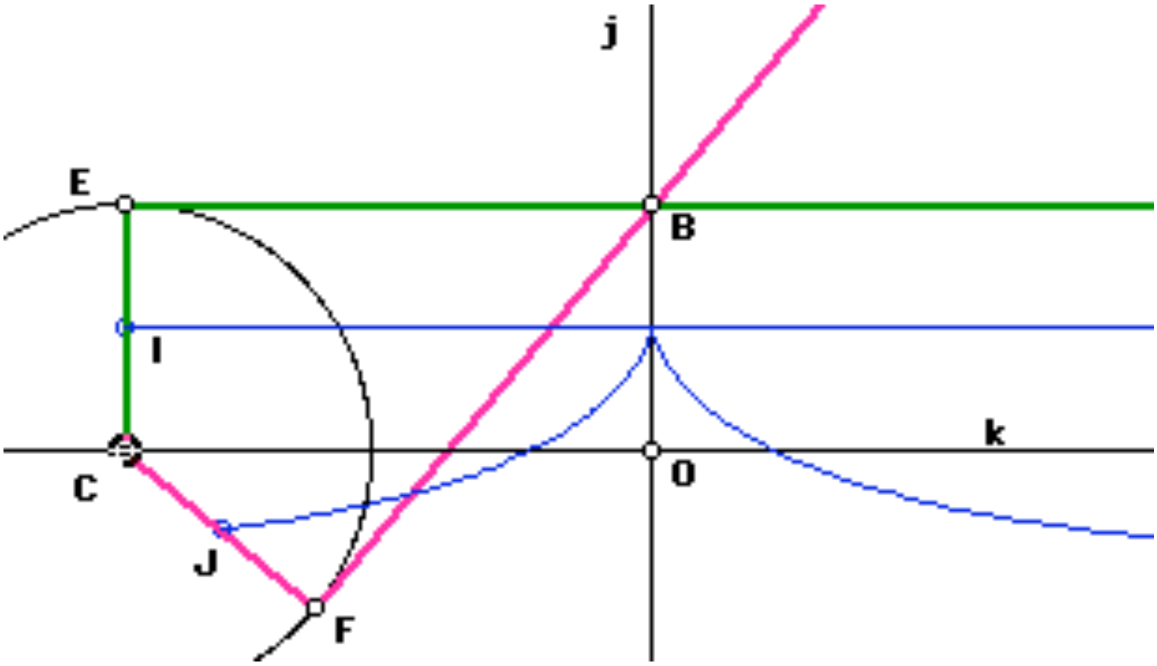
By some modern common accounts (Morris Kline, Thomas L. Heath), here's how Diocles constructed the curve in his book

On Burning-Glasses:

Let AB and CD be perpendicular diameters of a circle. Let E be a point on arc[B,C], and Z be a point on arc[B,D], such that BE, BZ are equal. Draw ZH perpendicular to CD. Draw ED. Let P be intersection[ZH,ED]. The cissoid is the locus of all points P determined by all positions of E on arc[B,C] and Z on arc[B,D] with arc[B,E]=arc[B,Z]. (The portion of the curve that lies outside of the circle is a later generalization).

In the curve, we have CH/HZ=HZ/HD=HD/HP. Thus HZ and HD are two mean proportionals between CH and HP. Proof: taking CH/HZ=HZ/HD, we have $CH * HD = HZ^2$. triangle[D,C,Z] is a right triangle since it's a triangle on a circle with one side being the diameter (elementary geometry). We know an angle[D,C,Z] and one side distance[D,C], thus by trignometry of right angles, we can derive all lengths DZ, CZ, and HZ. Substituting the results of computation in $CH * HD = HZ^2$ results an identity. Similarly, we know length HP and find HZ/HD=HD/HP to be an identity.

4.3 Newton's Carpenter's Square and Tangent



Newton showed that Cissoid of Diocles and the right Strophoid can be generated by sliding a right triangle. The midpoint J of the edge CF draws the Cissoid, the vertex F the Strophopid. This method also easily proves the tangent construction.

Step-by-step description:

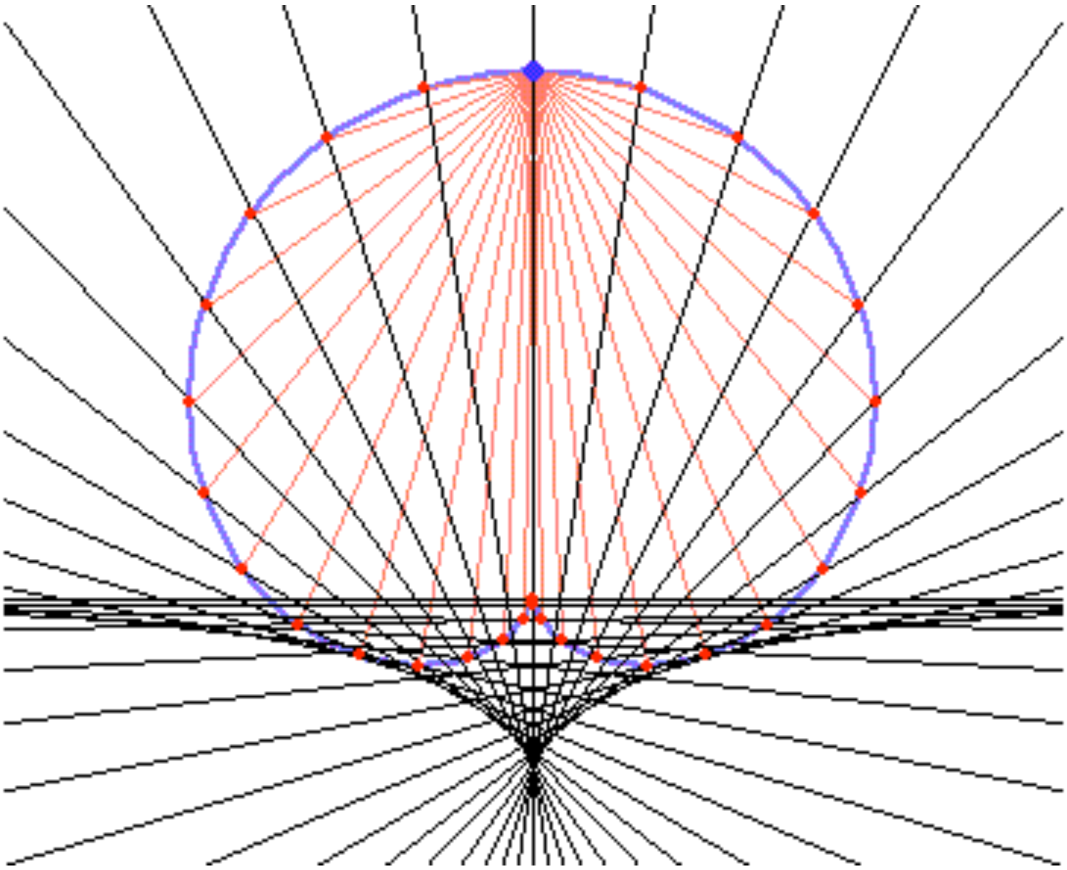
1. Let there be two distinct fixed points B and O, both on a given line j. (distance[B,O] will be the radius of the cissoid of Diocle we are about to construct.)
2. Let there be a line k passing O and perpendicular to j.
3. Let there be a circle centered on an arbitrary point C on k, with radius OB.

4. There are two tangents of this circle passing B, let the tangent points be E and F.
5. Let I be the midpoint between E and the center of the circle. Similarly, let J be the midpoint between F and the center of the circle.
6. The locus of I and J (as C moves on k) is the cissoid of Diocles and a line. Also, the locus of E and F is the right strophoid.

Tangent construction for Cissoid and Strophoid: Think of triangle[C,F,B] as a rigid moving body. The point C moves in the direction of vector[O,C], and point B moves in the direction of vector[B,F]. The intersection H (not shown) of normals of line[O,C] and line[B,F] is its center of rotation. J is the point tracing the Cissoid and is also a point on the triangle, thus HJ is normal to the Cissoid. For the Strophoid change the last sentence: Since the tracing point F is a point on the triangle, thus HF is normal to the Strophoid.
In 3D-XploreMath, this construction is shown automatically when Cissoid is chosen from the Plane Curve menu, just after the curve is drawn (or when it is redrawn by choosing Create from the Action menu or typing Command-K). In the Action Menu switch between Cissoid and Strophoid. Hold down the option key to slow the animation, hold down Control to reverse direction, and press the spacebar to pause.
In the animation, the tangent and normal are shown as

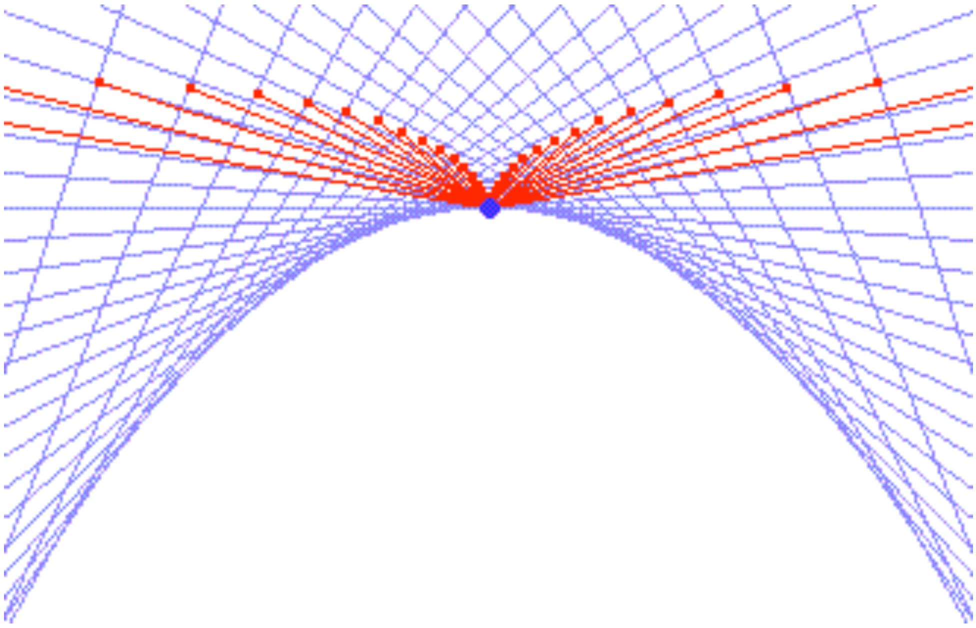
blue. The line from the critical point, from the so called momentary fixed point of the motion, is normal to the curve. This point is the intersection of the green lines; one of them is a vertical drop, the other perpendicular to the red line.

4.4 Pedal and Cardioid



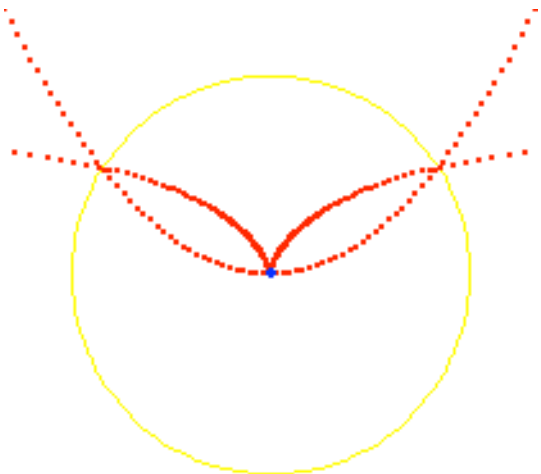
The pedal of a cissoid of Diocles with respect to a point P is the cardioid. If the cissoid's asymptote is the line y = 1 and its cusp is at the origin, then P is at $\{0, 4\}$. It follows by definition, the negative pedal of a cardioid with respect to a point opposite its cusp is the cissoid of Diocles.

4.5 Negative Pedal and Parabola



The pedal of a parabola with respect to its vertex is the cissoid of Diocles. (and then by definition, the negative pedal of a cissoid of Diocles with respect to its cusp is a parabola.)

4.6 Inversion and Parabola



The inversion of a cissoid of Diocles at cusp is a parabola.

4.7 Roulette of a Parabola

Let there be a fixed parabola. Let there be an equal parabola that rolls on the given parabola in such way that the two parabolas are symmetric to the line of tangency. The vertex of the rolling parabola traces a cissoid of Diocles.

XL.

Conchoid *

3D-XplorMath parametrization:

$$r = \frac{bb}{\cos t} + aa, \;\; x = r \cdot \cos t, \;\; y = r \cdot \sin t.$$

History

According to common modern accounts, the conchoid of Nicomedes was first conceived around 200 B.C by Nicomedes, to solve the angle trisection problem. The name conchoid is derived from Greek meaning “shell”, as in the word conch. The curve is also known as cochloid.

From E. H. Lockwood (1961):

> The invention of the ‘mussel-shell shaped’ conchoid is ascribed to Nicomedes (second century B.C.) by Pappus and other classical authors; it was a favorite with the mathematicians of the seventeenth century as a specimen for the new method of analytical geometry and calculus. It could be used (as was the purpose of its invention) to solve the two problems of doubling the cube and of trisecting an angle; and hence for every cubic or quartic problem. For this reason, Newton suggested that it should be treated as a ‘standard’ curve.

* This file is from the 3D-XplorMath project. Please see:
http://3D-XplorMath.org/

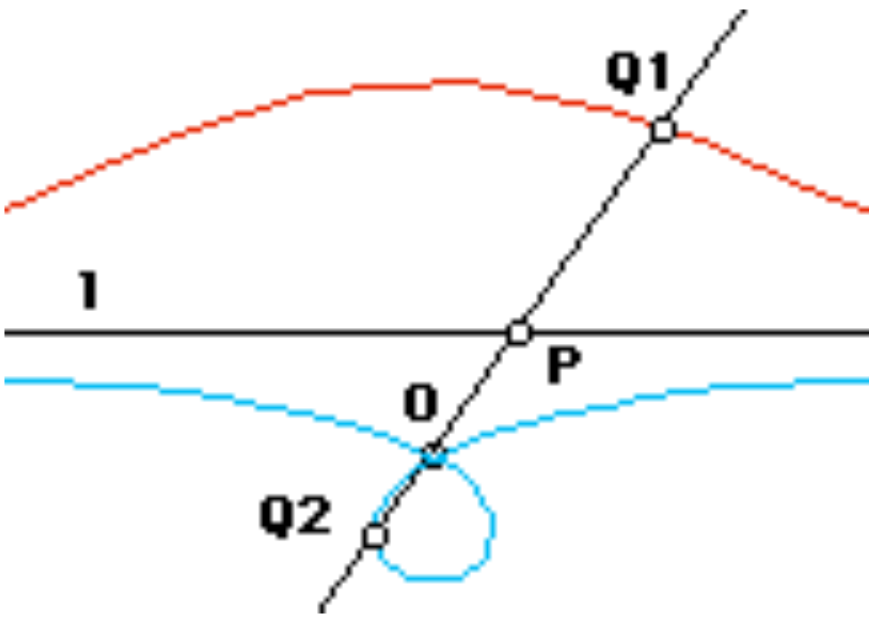
Description

The Conchoid of Nicomedes is a one parameter family of curves. They are special cases of a more general conchoid construction, being the conchoids of a line.

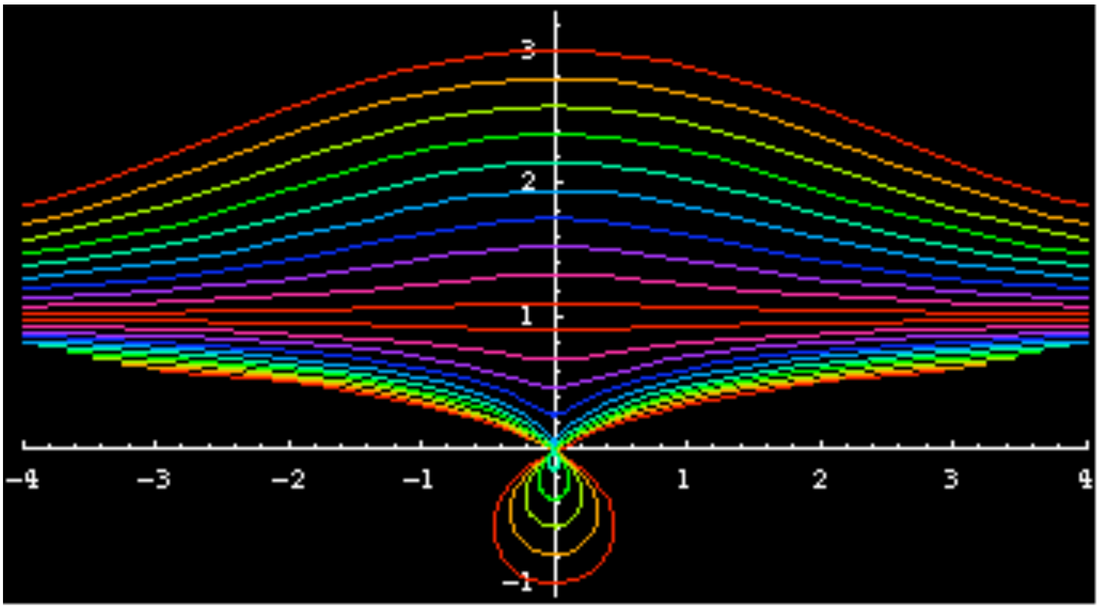
Step-by-step explanation:

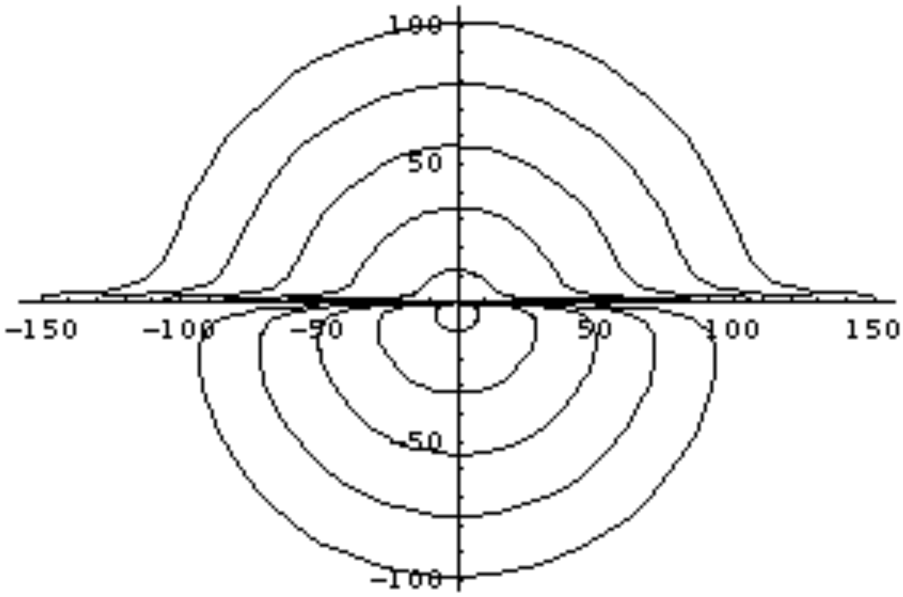
1. Given a line ℓ, a point O not on ℓ, and a distance k.
2. Draw a line m through O and any point P on ℓ.
3. Mark points $Q1$ and $Q2$ on m such that
$$\text{distance}[Q1, P] = \text{distance}[Q2, p] = k.$$
4. The locus of $Q1$ and $Q2$ as P varies on ℓ is the conchoid of Nicomedes.

The point O is called the *pole* of the conchoid, and the line ℓ is called its *directrix*. It is an asymptote of the curve.



The following figures shows the curve family. The pole is taken to be at the origin, and directrix is $y = 1$. The figure on top has constants k from -2 to 2. The one below has constants k from -100 to 100.





Formulas

Let the distance between pole and line be b, and the given constant be k. The curve has only the one parameter k, because for a given b, all families of the curve can be generated by varying k (they differ only in scale). (Similarly, we could use b as the parameter.) In a mathematical context, we should just use $b = 1$, however, it is convenient to have formulas that have both b and k. Also, for a given k, the curve has two branches. In a mathematical context, it would be better to define the curve with a signed constant k corresponding to a curve of only one branch. We will be using this intepretation of k. In this respect, the conchoid of Nichomedes is then two conchoids of a line with constants k and $-k$.

The curve with negative offset can be classified into three types: if $b < k$ there is a loop; if $b = k$, a cusp; and if $b > k$, it is smoothly imbedded. Curves with positive offsets are always smooth.

The following are the formulas for a conchoid of a line $y = b$, with pole O at the origin, and offset k.

Polar: $r = b/\sin(\theta) + k$, $-\pi/2 < \theta < \pi/2$.

This equation is easily derived: the line $x = b$ in polar equation is $r = b/\cos\theta$, therefore the polar equation is $r = b/\cos(\theta) + k$ with $-\pi/2 < \theta < \pi/2$ for a signed k (i.e., describing one branch.). Properties of cosine show that as θ goes from 0 to 2π, two conchoids with offset $\pm|k|$ results from a single equation $r = b/\cos(\theta) + k$. To rotate the graph by $\pi/2$, we replace cosine by sine.

Parametric:

$$(t + (kt)/\sqrt{b^2 + t^2}, b + (bk)/\sqrt{b^2 + t^2}),\ -\infty < t < \infty.$$

If we replace t in the above parametric equation by $b\tan(t)$, we get the form:

$$(k + \frac{b}{\cos(t)}) \cdot (\sin(t), \cos(t)), \ \frac{-\pi}{2} < t < \frac{3\pi}{2}, \ t \neq \frac{\pi}{2}.$$

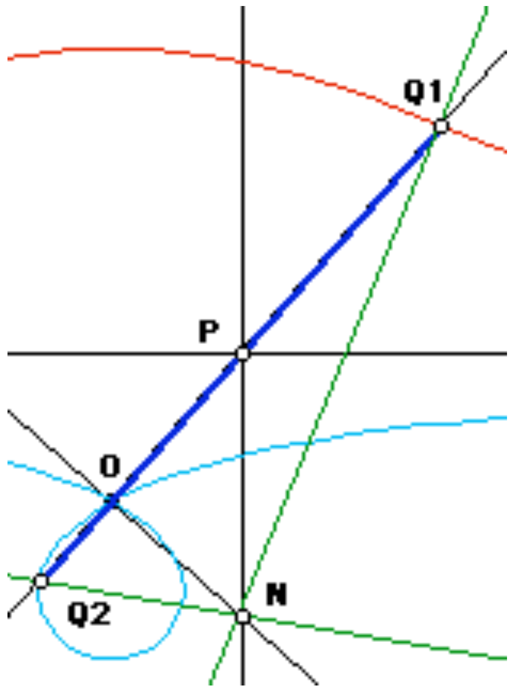
For conchoids of a line with positive and negative offsets k and pole at the origin, we have the quartic

Implicit Cartesian equation: $(x^2 + y^2)(y - b)^2 = k^2 y^2$.

If $k < b$, the point at the origin is an isolated point.
If $k < 0$ and $b < |k|$, the conchoid has a loop with area $(b\sqrt{k^2 - b^2} - 2bk \ln((k + \sqrt{k^2 - b^2})/b) + k^2 \arccos(b/k))$. The area between any conchoid of a line and its asymptote is infinite.

Tangent Construction

Look at the conchoid tracing as a mechanical device, where a bar line $[O, P]$ slides on a line at P and a fixed joint O. The point P on the bar moves along the directrix, and the point at O moves in the direction of the vector $[O, P]$. We know the direction of motion of the bar at the points O and P at arbitrary time.



The intersection of normals to these directions form the instantaneous center of rotation N. Since the tracing points

$Q1$ and $Q2$ are parts of the apparatus, N is also their center of rotation and therefore line $[N, Q1]$ and line $[N, Q2]$ are the curve's normals.

Angle Trisection

The curve can be used to solve the Greek Angle Trisection problem. Given an acute angle AOB, we want to construct an angle that is $1/3$ of AOB, with the help of the conchoid of Nicomedes.

Steps: Draw a line m intersecting segment $[A, O]$ and perpendicular to it. Let D be intersection of m and the line $[A, O]$, L the intersection of m and the line $[B, O]$. Suppose we are given a conchoid of Nicomedes, with pole at O, directrix m, and offset $2 \cdot \text{distance}[O, L]$. Draw a line ℓ intersecting L and perpendicular to m. Let C be an intersection of the curve and ℓ, the one on the opposite side of the pole.

Theorem. $\text{angle}[A, O, B] = 3 \cdot \text{angle}[A, O, C]$.

Proof:
$\measuredangle[A, O, C] = \measuredangle[O, C, L]$ because the line $[O, C]$ cuts parallel lines. Let:
q be the line $[O, C]$,
$N := m \cap q$,
M midpoint of $[N, C]$,
$k :=$ distance$[O, L]$.
By our construction,
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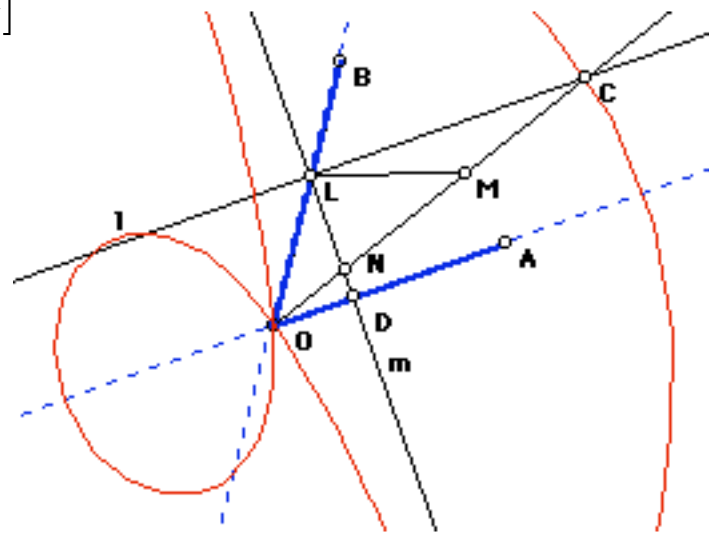


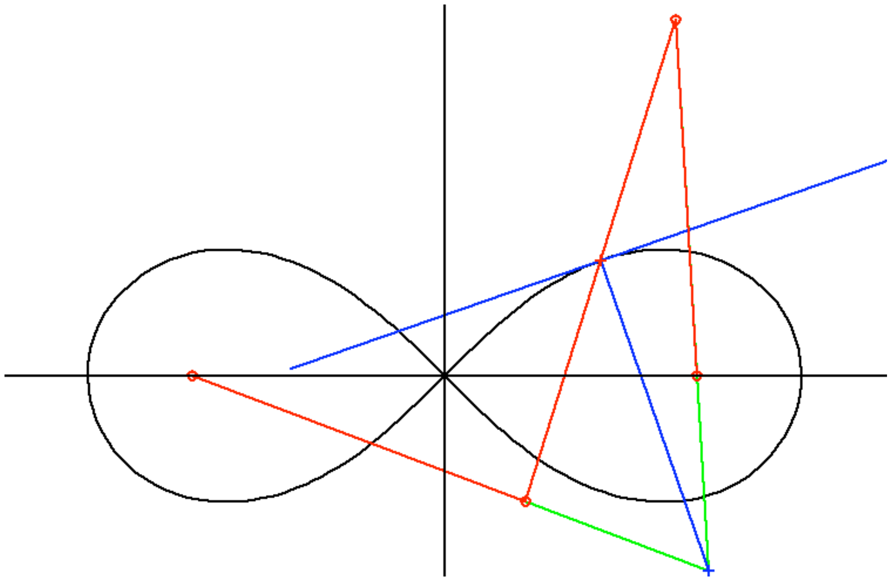
distance$[N, M]$ = distance$[M, C] = k$. Since NLC is a right triangle, we see that MN, ML, MC, and OL all have the same length, thus triangle $[M, L, C]$ and triangle $[M, L, N]$ are isosceles, and it follows that $\measuredangle[N, M, L] = 2 \cdot \measuredangle[M, C, L]$. Since distance$[O, L]$ = distance$[M, L]$, triangle $[M, L, O]$ is also isosceles, and thus its two base angles are equal. This shows that an angle equal to $\measuredangle[A, O, C]$ is 1/3 an angle equal to $\measuredangle[A, O, B]$.

The essential point where the conchoid makes the trisection possible is in the construction of the point C on ℓ such that distance$[N, C] = 2$ distance$[O, L]$, where N is the intersection of m and the line $[O, C]$. Note that for each new angle to trisect, a new conchoid is needed. This is in contrast to some other trisectrixes such as the quadratrix, where all angles can be trisected once the curve is given.

The conchoid can also be used to solve the classic problem of doubling the cube.

XL.

Lemniscate *



Parametrized family in 3DXM: $r := aa/(1+\sin^2(t))$,

$$x := r\cos(t)\big(1+(1-\tan(bb))\cos(t)/2\big),$$
$$y := r\sin(t)\big(\cos(t)+1-\tan(bb)\big).$$

The Lemniscate is a figure-eight curve with a simple **mechanical construction** attributed to Bernoulli: Choose two 'focal' points F_1, F_2 at distance L, then take three rods, one of length L, two of length $L/\sqrt{2}$. The short ones can rotate around the focal points and they are connected by the long one with joints which allow rotation. This machine has one degree of freedom and the *midpoint* of the long rod traces out the Lemniscate when one of the short rods is rotated. *Other drawing pens* can be chosen: set $bb \neq \pi/4$, see the default `Morph` (it scales $aa = aa(bb)$). Mechanical constructions of curves often come with simple **tangent constructions**. We imagine that a plane is

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attached to the long rod. Then every point of this plane traces out a curve when the rods move. At each moment the endpoints of the long rod move orthogonal to the short rods (namely on circles around the focal points). This says that the straight extensions of the short rods (green) intersect in the momentary center of rotation. At this moment every point of the plane rotates around this center so that the tangent of each point's orbit is orthogonal to its connection with the momentary center of rotation. (Compare the other mechanically constructed curves.)
The Lemniscate has the implicit equation:

$$(x^2 + y^2)^2 = x^2 - y^2.$$

Divide this by $r^2 := x^2 + y^2$ to get the polar form:

$$r^2 = \cos(\phi)^2 - \sin(\phi)^2.$$

Parametrizations are not unique, here is one:

$$x(t) := \cos(t)/(1 + \sin(t)^2)$$
$$y(t) := \sin(t) \cdot \cos(t)/(1 + \sin(t)^2).$$

The points $F_1, F_2 := \pm 1/\sqrt{2}$ are called Focal points of the Lemniscate because of the special property:

$$|P - F_1| \cdot |P - F_2| = |F_1 - F_2|^2/4.$$

If one takes the complex square root of a circle which touches the y-axis from the right at 0 then one also obtains a Lemniscate. In the Conformal Category, choose $z \to \sqrt{z}$, and then in the Action Menu, select Choose Circle by Mouse, and create a circle that is tangent to the y-axis at 0.)

The inversion map: $(x, y) \mapsto (x, y)/(x^2 + y^2)$ often transforms some interesting curve into another interesting curve. And indeed, the Lemniscate, with the above parametrization, is transformed by inversion into the curve

$$x = 1/\cos(t),\ y = \sin(t)/\cos(t).$$

Since $x^2 - y^2 = 1$, this is a hyperbola, so we could also have obtained the Lemniscate from the standard hyperbola by inversion.

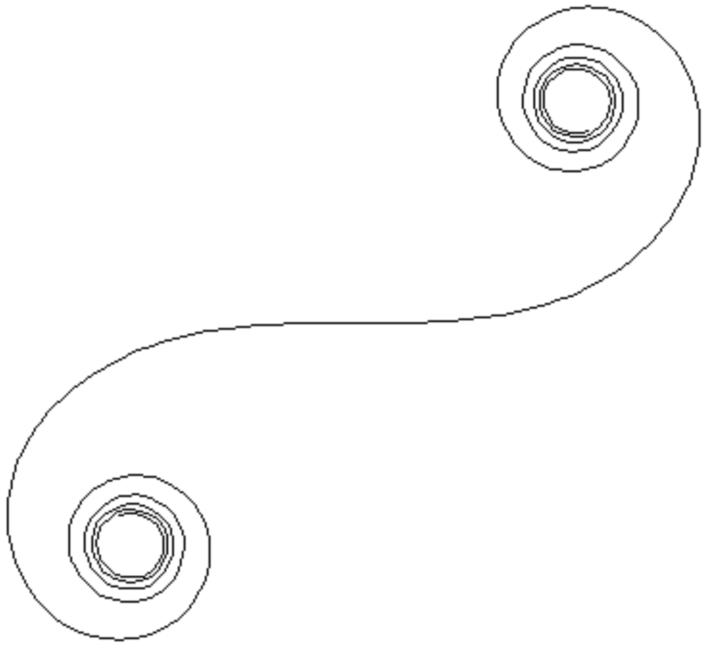
We note that not every figure-eight curve is a Lemniscate, another figure-eight is obtained by the simpler parametrization:

$$x(t) := \cos(t)$$
$$y(t) := \sin(t) \cdot \cos(t),$$

which has the implicit equation $y^2 = x^2(1 - x^2)$.

H.K.

Clothoid *



The Clothoid, also called Spiral of Cornu, is a curve whose curvature is equal to its arclength. It has the parametric formula:

$$\left(\int_0^t \cos(x^2/2)\,dx, \int_0^t \sin(x^2/2)\,dx\right).$$

Discussion

If a plane curve is given by a parametric formula $(f(t), g(t))$, then the length of the part corresponding to a parameter interval $[a, t]$ is $s(t) = \int_a^t \sqrt{f'(\tau)^2 + g'(\tau)^2}\,d\tau$. If we apply this formula to the Clothoid we see that the arclength corresponding to the interval $[0, t]$ is $s(t) = \int_0^t 1\,dt = t$, so

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that the parameter t is precisely the (signed!) arclength measured along the curve from its midpoint, $(0, 0)$.

Next, recall that the curvature κ of a plane curve is defined as the rate of change (with respect to arclength) of the angle θ that its tangent makes with some fixed line (which we can take to be the x-axis). And since the slope $\frac{dy}{dx}$ of the curve is $\tan(\theta)$, and by the chain rule $\frac{dy}{dx} = (dy/dt)/(dx/dt) = \frac{g'}{f'}$, we see that $\theta(t) = \arctan(g'(t)/f'(t))$. So if we assume that parameter t is arclength, then using the formulas for the derivative of the arctangent and of a quotient, we see that:

$$\kappa(t) = \theta'(t) = -g'(t)f''(t) + f'(t)g''(t),$$

(where we have ignored the denominator, since parameterization by arclength implies that it equals unity). Applying this to the Clothoid, we obtain $\kappa(t) = t$. Since the arclength function is also t, this shows that the Clothoid is indeed a curve whose curvature function is equal to its arclength function.

The Fundamental Theorem of Plane Curves

Next let's look at this question from the other direction, and also more generally. Suppose we are given a function $\kappa(t)$. Can we find a plane curve parameterized by arclength $(f(t), g(t))$ such that κ is its curvature function? Recall from above that $\frac{d\theta}{dt} = \kappa$, and of course $\frac{dx}{dt} = f'(t)$ and $\frac{dy}{dt} = g'(t)$. Now, since $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 1$, while $\frac{dy}{dt} / \frac{dx}{dt} = dy/dx = \tan(\theta)$, it follows from elementary trigonometry that $\frac{dx}{dt} = \cos(\theta)$ while $\frac{dy}{dt} = \sin(\theta)$. Thus we have the following system of three differential equations for the three functions $\theta(t)$, $f(t)$, and $g(t)$:

$$\theta'(t) = \kappa(t), \quad f'(t) = \cos(\theta(t)), \quad g'(t) = \sin(\theta(t)).$$

The first equation is solved by $\theta(\tau) = \theta_0 + \int_0^\tau \kappa(\sigma)\, d\sigma$, and substituting this in the other two equations, we find that the general solutions for f and g are given by:

$$f(t) = x_0 + \int_0^t \cos(\theta_0 + \int_0^\tau \kappa(\sigma)\, d\sigma)\, d\tau$$
$$g(t) = y_0 + \int_0^t \sin(\theta_0 + \int_0^\tau \kappa(\sigma)\, d\sigma)\, d\tau.$$

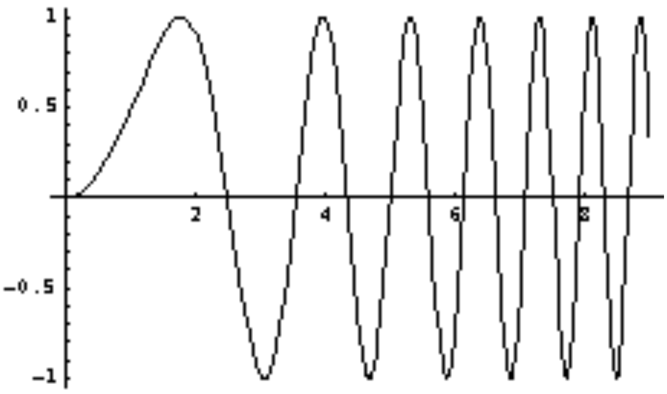
This is an elegant explicit solution to our question! It shows that not only is there a solution to our question (say the one obtained by setting x_0, y_0 and θ_0 all equal to zero), but also that the solution is unique up to a translation (by (x_0, y_0)) and a rotation (by θ_0), that is unique up to a general rigid motion.

This fact has a name—it is called The Fundamental Theorem of Plane Curves. It tells us us that most geometric and most economical descriptions of plane curves is not via parametric equations, which have a lot of redundancy, but rather by the single function κ that gives the curvature as a function of arclength.

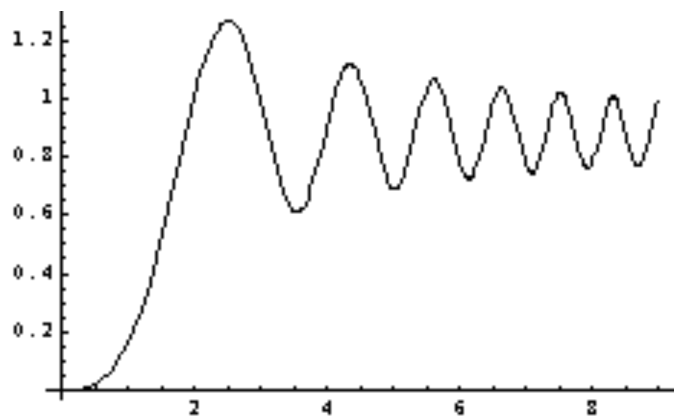
Exercise Take $\kappa(t) = t$ and check that the above formulas give the parametric equations for the Clothoid in this case.

Back to the Clothoid

We close with a few more details about the Clothoid. First, here is a plot of the integrand $\sin(x^2/2)$:



and next a plot of its indefinite integral, $\int_0^t \sin(x^2/2)\,dx$, the so-called Fresnel integral:

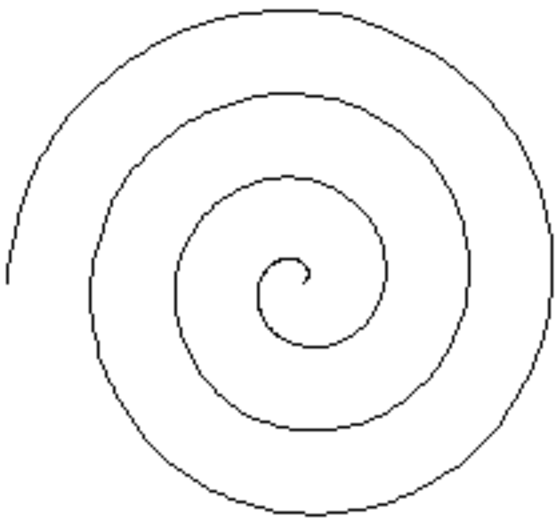


From this plot we see that the y-coordinate oscillates. Its limit as t goes to infinity is $\sqrt{\pi}/2$, from which we see that the centers of the two spirals of the Clothoid are at $\pm(\sqrt{\pi}/2, \sqrt{\pi}/2)$.

XL & RSP.

Archimedean Spirals *

An Archimedean Spiral is a curve defined by a polar equation of the form $r = \theta^a$. Special names are being given for certain values of a. For example if $a = 1$, so $r = \theta$, then it is called Archimedes' Spiral.

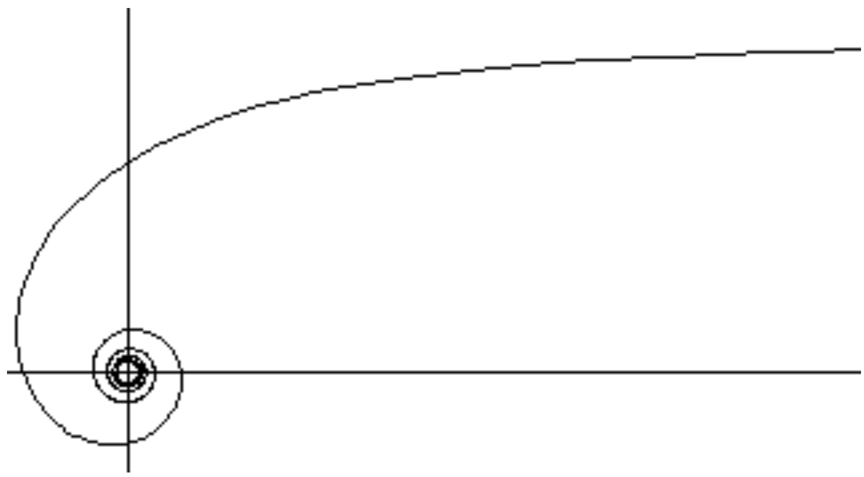


Archimede's Spiral

Formulas in 3DXM:

$r(t) := t^{aa}$, $\theta(t) := t$,
Default Morph:
$$-1 \leq aa \leq 1.25.$$

For $a = -1$, so $r = 1/\theta$, we get the reciprocal (or hyperbolic) spiral:



Reciprocal Spiral

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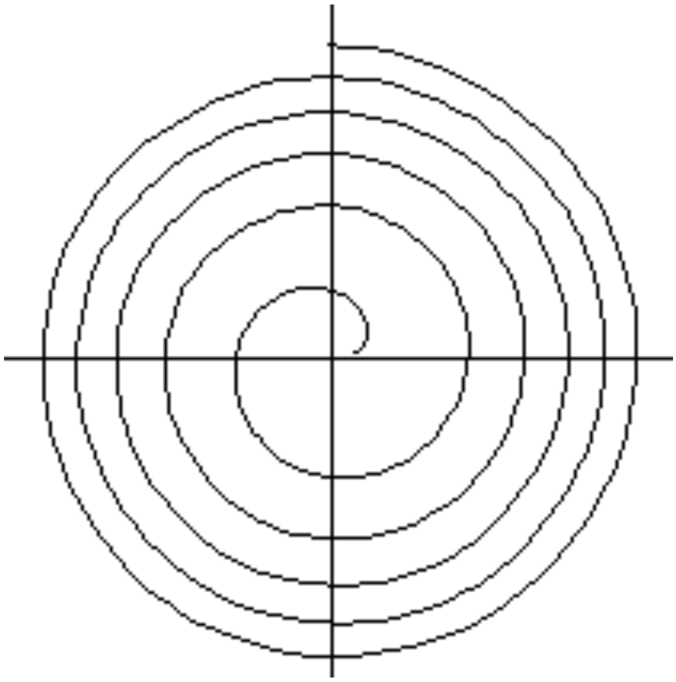
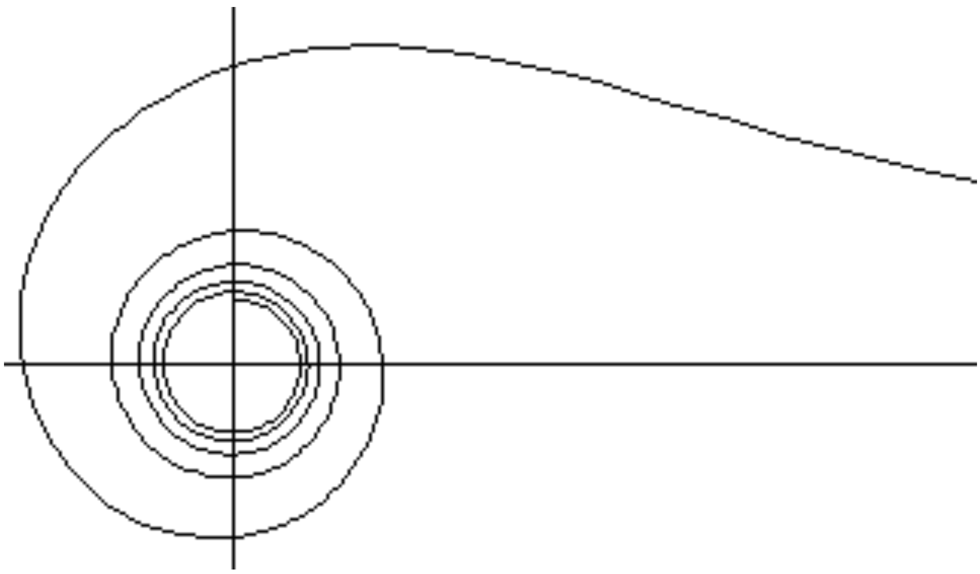
The case $a = 1/2$, so $r = \sqrt{\theta}$, is called the Fermat (or hyperbolic) spiral.



Fermat's Spiral

While $a = -1/2$, or $r = 1/\sqrt{\theta}$, it is called the Lituus:



Lituus

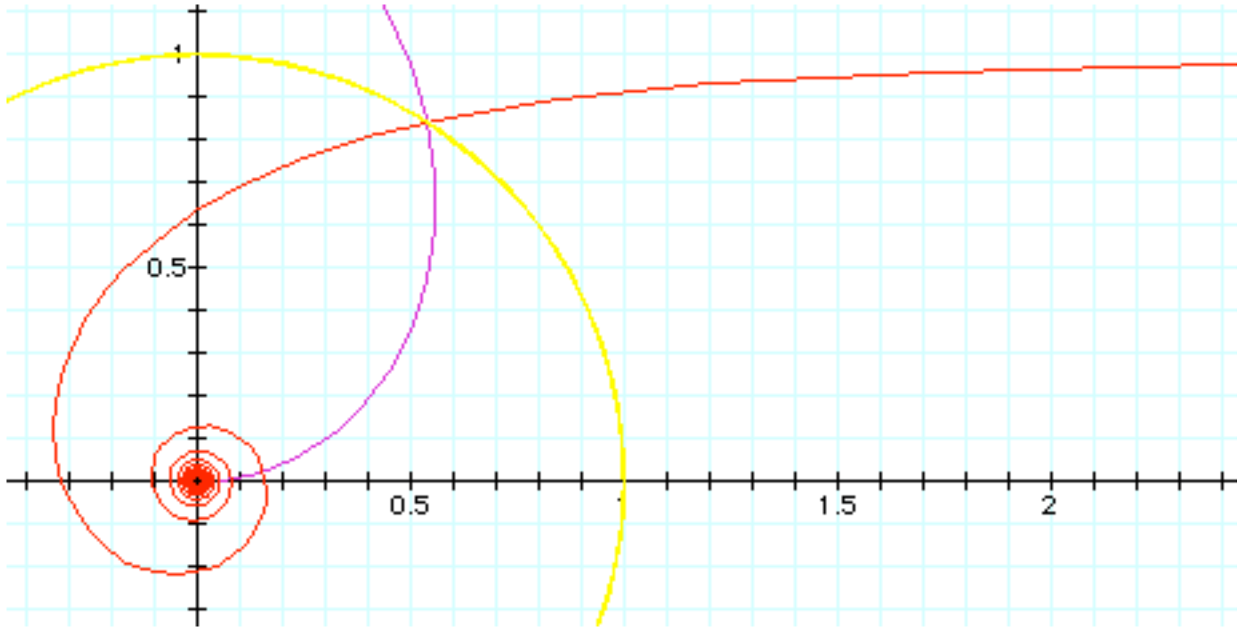
In 3D-XplorMath, you can change the parameter a by going to the menu Settings $\rightarrow$ Set Parameters, and change the value of aa. You can see an animation of Archimedean spirals where the exponent $a = aa$ varies gradually, between -1 and 1.25. See the *Animate Menu*, entry *Morph*.

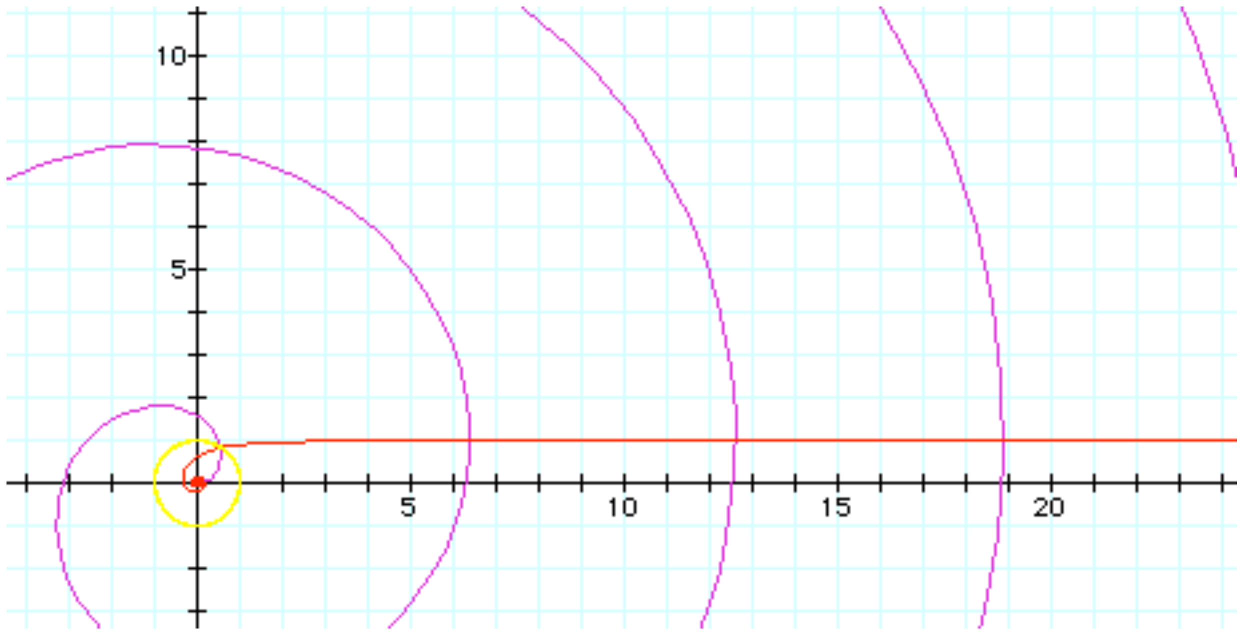
The reason that the parabolic spiral and the hyperbolic spiral are so named is that their equations in polar coordinates, $r\theta = 1$ and $r^2 = \theta$, respectively resembles the equations for a hyperbola ($xy = 1$) and parabola ($x^2 = y$) in rectangular coordinates.

The hyperbolic spiral is also called reciprocal spiral because it is the inverse curve of Archimedes' spiral, with inversion center at the origin.

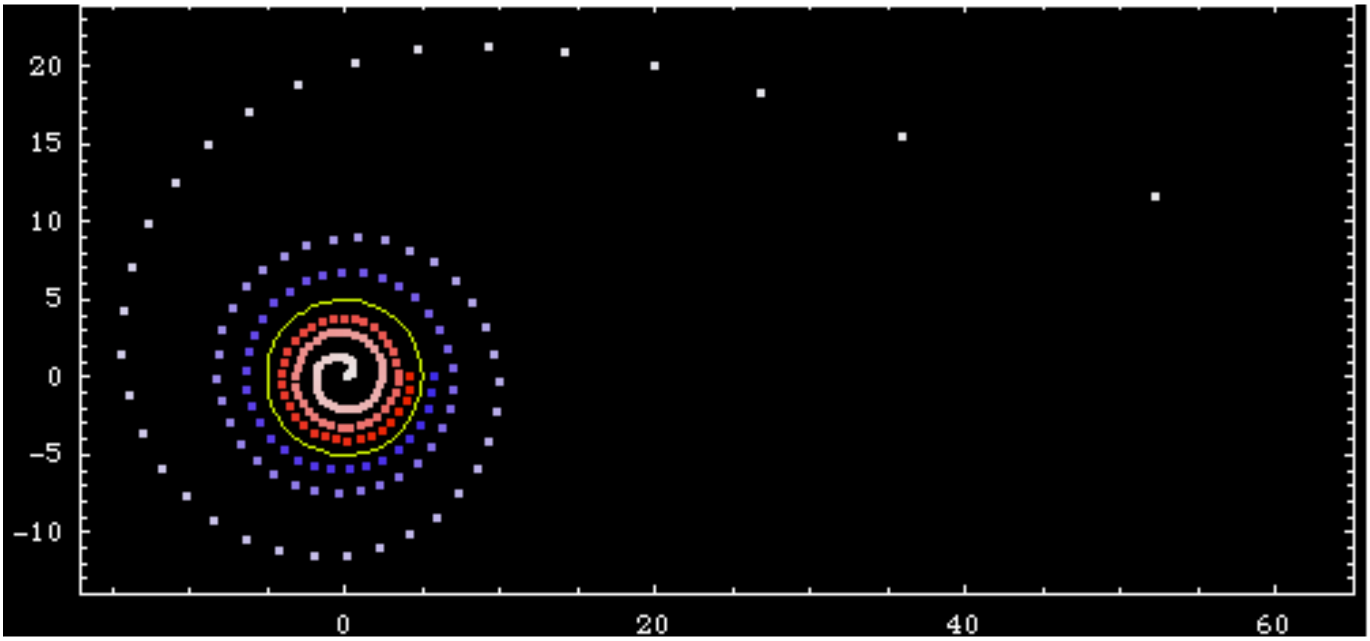
The inversion curve of any Archimedean spirals with respect to a circle as center is another Archimedean spiral, scaled by the square of the radius of the circle. This is easily seen as follows. If a point P in the plane has polar coordinates (r, θ), then under inversion in the circle of radius b centered at the origin, it gets mapped to the point P' with polar coordinates $(b^2/r, \theta)$, so that points having polar coordinates (t^a, θ) are mapped to points having polar coordinates $(b^2 t^{-a}, \theta)$.
From the above, we can see that the Archimedes' spiral inverts to the reciprocal spiral, and Fermat's spiral inverts to the Lituus.

The following two images illustrates Archimedes's spiral and Reciprocal spiral as mutual inverses. The red curve is the reciprocal spiral, the purple is the Archimedes' spiral. The yellow is the inversion circle.





The following image illustrates a Lituus and Fermat's spiral as mutual inverses. The red curve is the Fermat's spiral. The blue curve is its inversion, which is a lituus scaled by 5^2. The yellow circle is the inversion circle with radius 5. Note that points inside the circle gets mapped to outside of the circle. The closer the point is to the origin, the farther is its corresponding point outside the circle.



XL.

The Logarithmic Spiral*

The parametric equations for the Logarithmic Spiral are:

$$x(t) = aa * \exp(bb\ t)\cos(t)$$
$$y(t) = aa * \ exp(bb\ t)\sin(t)$$

This spiral is connected with the complex exponential as follows:

$$x(t) + i\,y(t) = aa \exp((bb + i)t).$$

The animation that is automatically displayed when you select Logarithmic Spiral from the Plane Curves menu shows the osculating circles of the spiral. This illustrates an interesting theorem, namely if the curvature is a monotone function along a segment of a plane curve, then the osculating circles are nested. (See page 31 of J.J. Stoker's "Differential Geometry", Wiley-Interscience, 1969).

For the logarithmic spiral this implies that the plane minus the origin is foliated by its osculating circles.

WHAT!? Hey! Wait a minute! If a smooth manifold M of dimension n is foliated by leaves of dimension k, and if S is a k-dimensional connected submanifold of M such that S is tangent at every point to one of the leaves, then S is an

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http://3D-XplorMath.org/

open subset of a leaf. But taking M to be the punctured plane, and S the logarithmic spiral, the osculating circle foliation gives a counterexample to this well-known theorem (which is little more than the definition of a leaf). This paradox was pointed out to me by Étienne Ghys. (Read words backwards below for the explanation.)

EHT NOITAILOF SLIAF OT EB
HTOOMS GNOLA EHT LARIPS

R.S.P

Cycloid*

Cycloids are generated by rolling a circle on a straight line and tracing out the path of some point along the radius. The parametric equation for such a cycloid is:

$$x(t) = aa \cdot t - bb \cdot \sin t$$
$$y(t) = aa - bb \cdot \cos t,$$

where aa is the radius of the rolling circle and bb is the distance of the drawing point from the center of the circle.

The choice $bb = aa$ gives the standard cycloid.

Cycloids have other cycloids of the same size as evolutes, see the Action Menu Entry *Show Osculating Circles with Normals*. This fact is responsible for Huyghen's cycloid pendulum having its period independent of the amplitude of the oscillation.

H.K.

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About Epicycloids and Hypocycloids *

See also the ATOs for Spherical Cycloids

Definition and tangent construction

Epicycloids resp. Hypocycloids are obtained if one circle of radius r rolls on the outside resp. inside of another circle of radius R.

In 3D-XplorMath: $r = hh,\ R = aa.$

The angular velocity of the rolling circle is fr times the angular velocity of the fixed circle (negative for hypocycloids). fr has to be an integer for the hypocycloid to be closed. The formulas do not actually roll one circle around another, they represent the curve as superposition of two rotations:

$$fr := (R - r)/(-r);$$
$$c.x := (R - r)\cos(t) + r\cos(fr \cdot t);$$
$$c.y := (R - r)\sin(t) + r\sin(fr \cdot t);$$

Double generation: If one changes the radius of the rolling circle from r to $R-r$ then these formulas are preserved, except for the parametrization speed. To view this in 3DXM replace hh by $aa - hh$.

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Epicycloids are obtained if one circle of radius $r = -hh$ rolls on the outside of another circle of radius $R = aa$. The angular velocity of the rolling circle is $fr > 0$ times the angular velocity of the fixed circle (again an integer for closed epicycloids).

$$fr := (R + r)/r;$$
$$c.x := (R + r)\cos(t) - r\cos(fr * t);$$
$$c.y := (R + r)\sin(t) - r\sin(fr * t);$$

These formulas agree with those of the hypocycloids except for the sign of r. We view them in 3DXM by using negative hh.

We can also use a drawing stick of length $ii * r$. The default morph shows this: $0.5 < ii < 1.5$.

These more general ($ii <> 1$) rolling curves were important for Greek astronomy because the planets orbit the sun (almost) on circles. Therefore, when one looks at other planets from earth, their orbits are (almost) such rolling curves. It is no surprise that many of these curves have individual names: Astroid, Cardioid, Limacon, Nephroid are examples in 3DXM.

Tangent construction.
Rolling curves have a very simple tangent construction. The point of the rolling circle which is in contact with the base curve has velocity zero – just watch cars going by. This means that the connecting segment from this point

of contact of the wheel to the endpoint of the drawing stick is the radius of the momentary rotation. The tangent of the curve which is drawn by the drawing stick is therefore orthogonal to this momentary radius.

The 3DXM-demo draws the rolling curve and shows its tangents.

H.K., R.S.P.

Cardioids and Limaçons*

Cardioids and Limaçons are obtained if on the outside of one fixed circle of radius $r = aa$ another circle of the same radius rolls. These curves are traced by a radial stick of length $R = ii * r$, $ii = 1$ for Cardioids and $ii > 1$ for Limaçons.

One choice of parametric equations for these curves is:

$$x(t) = 2r\cos(t) + R\cos(2t)$$
$$y(t) = 2r\sin(t) + R\sin(2t).$$

The evolute of the Cardioid is a smaller Cardioid, see in the Action Menu the entry `Show Osculating Circles with Normals`. In the entry `Add Caustics` one can rotate all normals by a fixed amount and these rotated lines always envelope a Cardioid.

To see the Cardioid generated by rolling a larger circle around a smaller one choose in the exhibit *Epi- and Hypocycloids* parameters $hh = 2 * aa, \ ii = 1$.

The image of the unit circle under the complex map

$$z \mapsto w(z) = z^2 + 2z$$

is a Cardioid; images of larger circles (around 0) are Limacons. Inverses $z \mapsto 1/w(z)$ of Limaçons are figure-eight shaped, including a Lemniscate.

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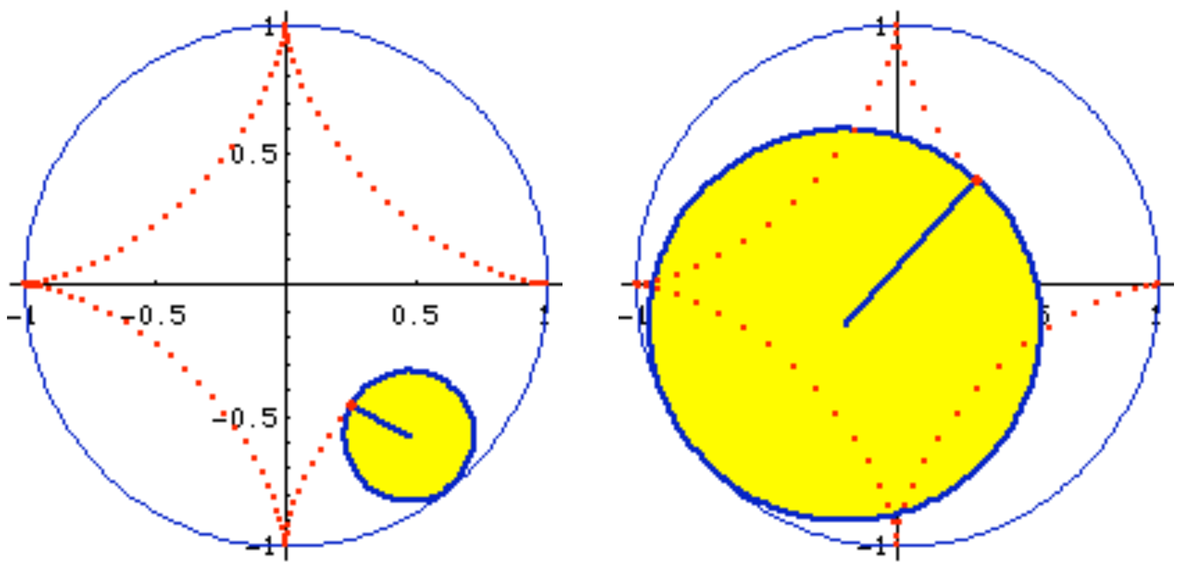
Astroid *

Parametrization in 3DXM: $c(t) := aa \cdot (\cos^3(t), \sin^3(t))$.

Implicit equation: $x^{2/3} + y^{2/3} = aa^{2/3}$.

Description



An Astroid is a curve traced out by a point on the circumference of one circle (of radius r) as that circle rolls without slipping on the inside of a second circle having four times or four-thirds times the radius of the first. The latter is known as *double generation*. The Astroid is thus a special kind of a hypocycloid—the family of analogous curves one

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gets if one allows the ratios of the radii to be arbitrary. In 3D-XplorMath, the radius r is represented by the parameter aa. A nice geometric property of the Astroid is that its tangents, when extended until they cut the x-axis and the y-axis, all have the same length. This means, if one leans a ladder (say of length L) against a wall at all possible angles, then the envelope of the ladder's positions is part of an Astroid. Since (by symmetry) the tangent to the Astroid at a point p closest to the origin has a slope of plus or minus one, it follows that the distance of p from the origin is $L/2$, and so L is the "waist-diameter" of the Astroid, i.e., the distance from p to $-p$. Since the diagonal of the Astroid clearly has length $2L$, it is twice as long as the waist-diameter.

It can be shown that the normals of an Astroid envelope an Astroid of twice the size. (To see a visual demonstration of this fact, in 3D-XplorMath, select Show Osculating Circles and Normals from the Action Menu.) If you think about what this means, you should see that it gives a ruler construction for the Astroid: Intersect each ladder (between the x-axis and the y-axis) for the smaller Astroid with the orthogonal and twice as long ladder (between the 45-degree lines) for the larger Astroid.

More Formulas

The initial formulas give an astroid centered at the origin with its four cusps lying on the axes at distance aa from the origin. To derive a polynomial equation first cube both sides of the above implicit equation (with $aa = 1$), factorize and simplify:

$$
\begin{aligned}
1 &= x^2 + y^2 + 3x^{4/3}y^{2/3} + 3x^{2/3}y^{4/3} \\
1 - x^2 - y^2 &= 3x^{2/3}y^{2/3}(x^{2/3} + y^{2/3}) = 3x^{2/3}y^{2/3}.
\end{aligned}
$$

Then cube again:

$$(1 - x^2 - y^2)^3 = 27x^2y^2.$$

History

Quote from Robert C. Yates, 1952:

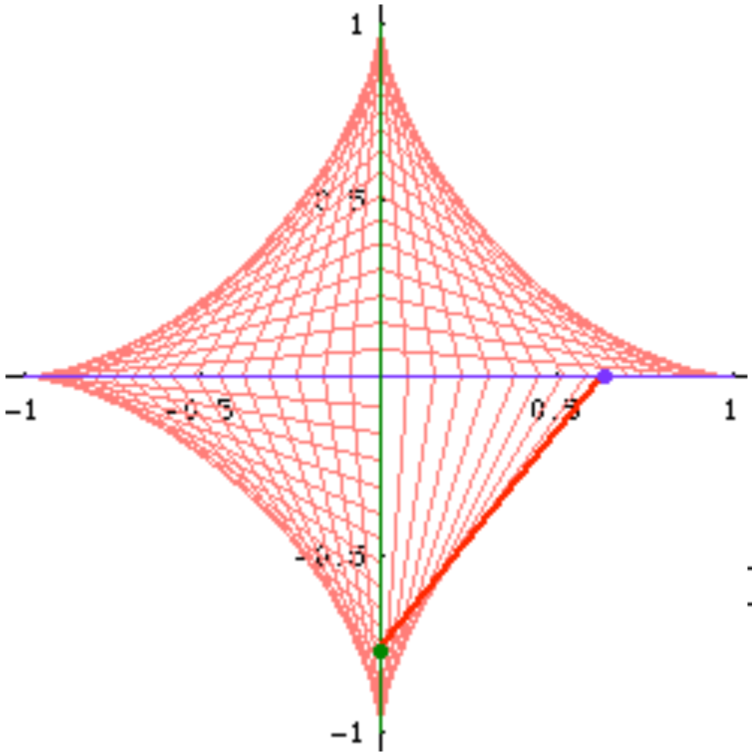
> The cycloidal curves, including the astroid, were discovered by Roemer (1674) in his search for the best form for gear teeth. Double generation was first noticed by Daniel Bernoulli in 1725.

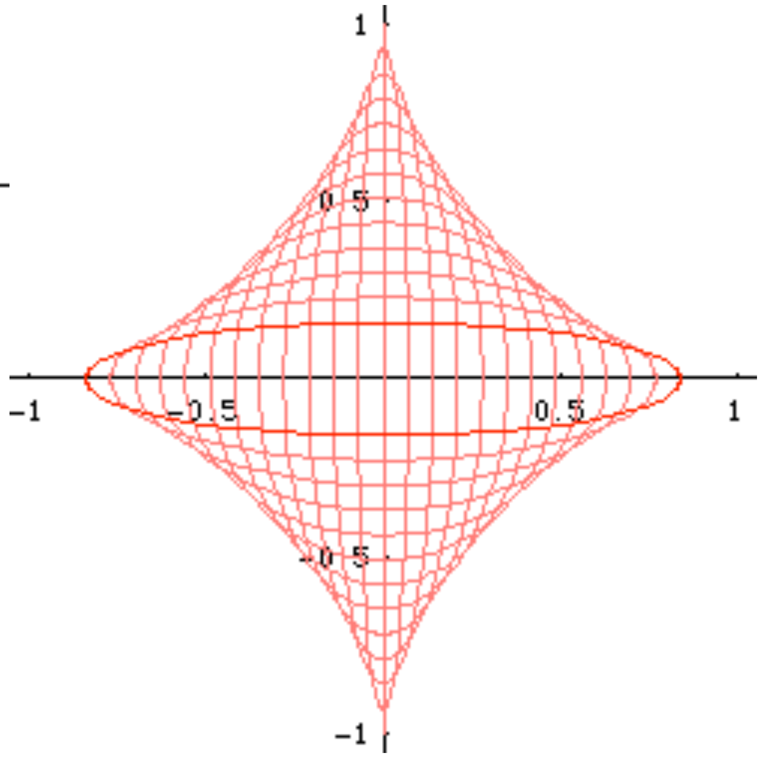
Quote from E. H. Lockwood, 1961:

> The astroid seems to have acquired its present name only in 1838, in a book published in Vienna; it went, even after that time, under various other names, such as cubocycloid, paracycle, four-cusp-curve, and so on. The equation $x^{2/3} + y^{2/3} = a^{2/3}$ can, however, be found in Leibniz's correspondence as early as 1715.

Properties

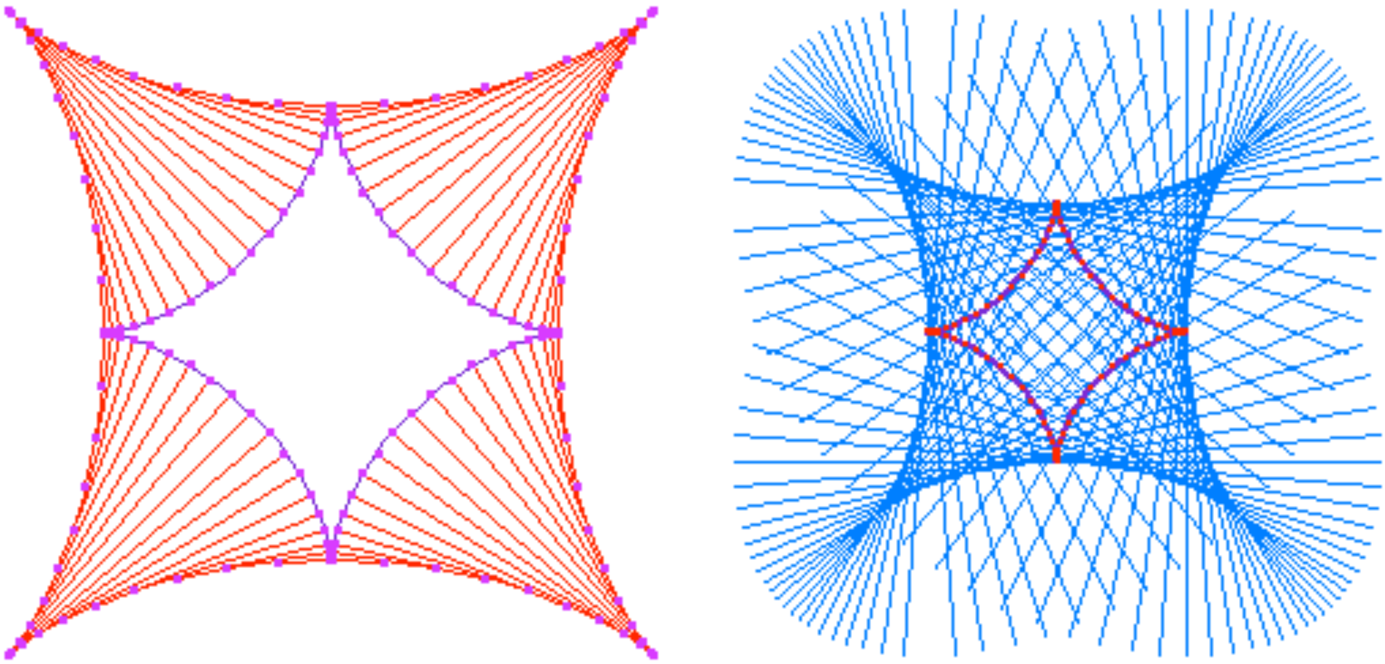
Trammel of Archimedes and Envelope of Ellipses
Define the *axes* of the astroid to be the two perpendicular lines passing through the pairs of alternate cusps. A fundamental property of the Astroid is that the length of the segment of a tangent between these two axes is a constant. The Trammel of Archimedes is a mechanical device that is based on this property: it has a fixed bar whose ends slide on two perpendicular tracks. The envelope of the moving bar is then the Astroid, while any particular point on the bar will trace out an ellipse.
The Astroid is also the envelope of co-axial ellipses whose sum of major and minor axes is constant.





The Evolute of the Astroid

The evolute of an astroid is another astroid. (In fact, the evolute of any epi- or hypo- cycloid is a scaled version of itself.) In the first figure below, each point on the curve is connected to the center of its osculating circle, while in the second, the evolute is seen as the envelope of normals.

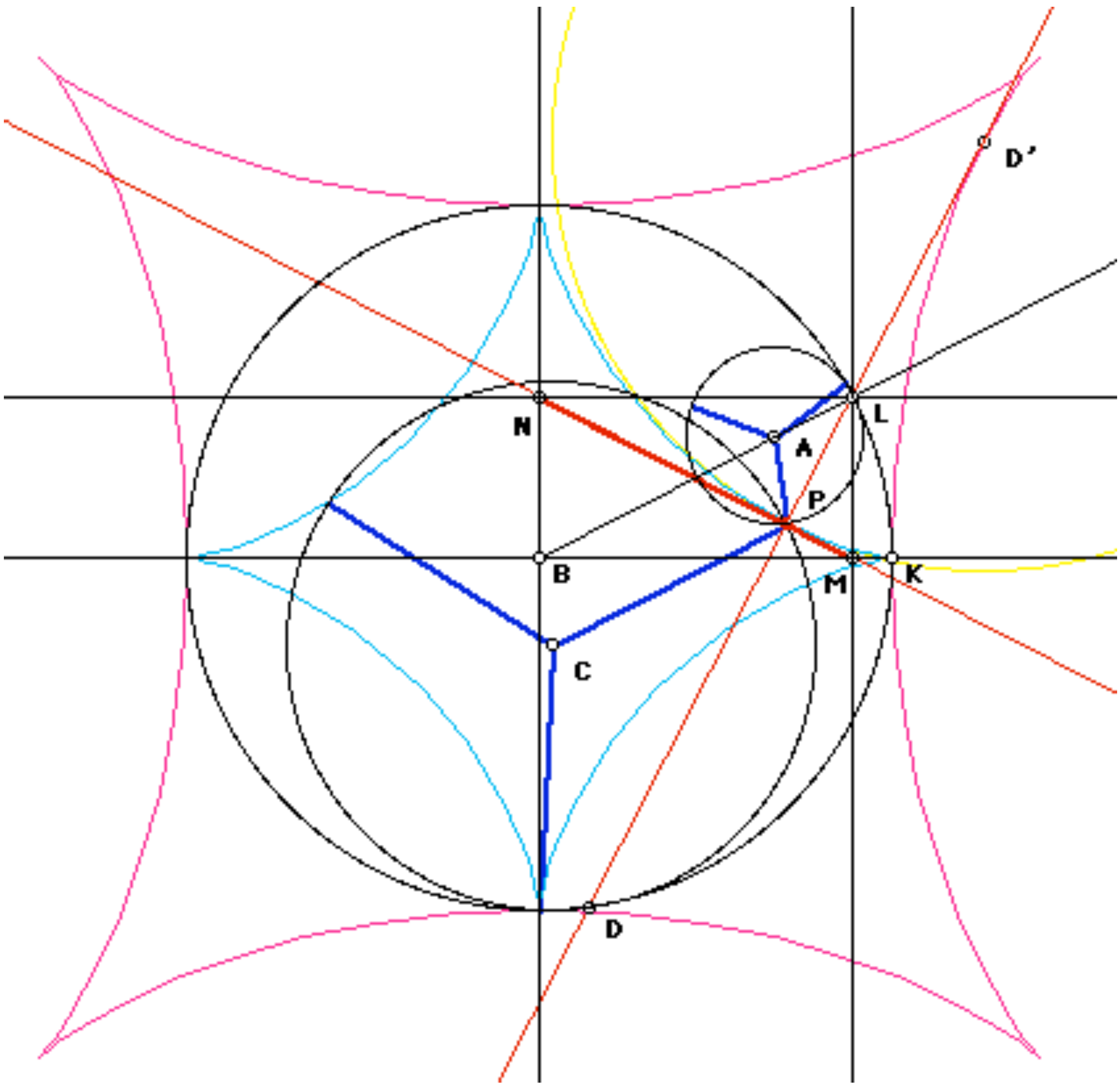


Curve Construction

The Astroid is rich in properties that can be used to devise other mechanical means to generate the curve and to construct its tangents, and the centers of its osculating circles.

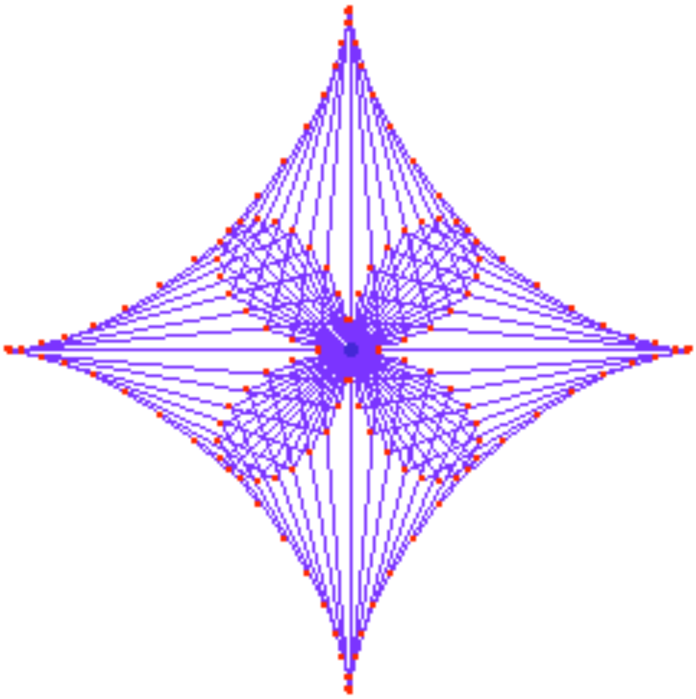
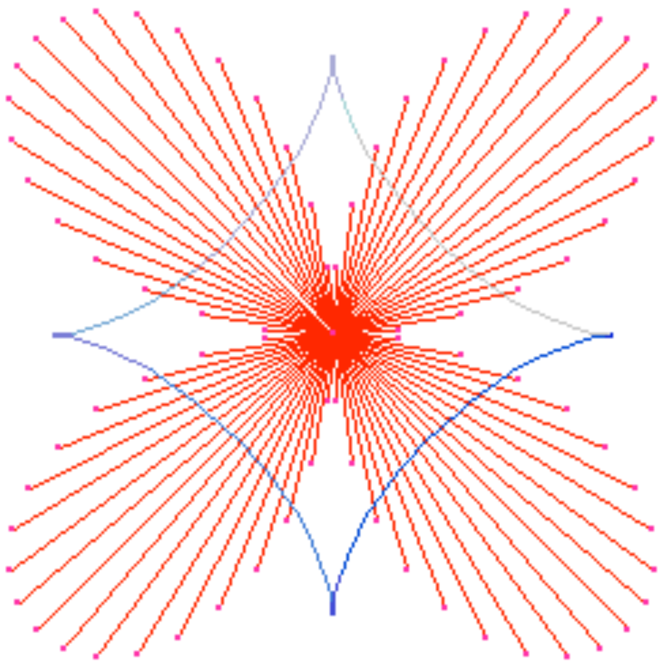
Suppose we have a circle C centered at B and passing through some point K. We will construct an Astroid that is also centered at B and that has one of its cusps at K.

Choose the origin of a cartesian coordinate system at B, and take the point $(1, 0)$ at K. Given a point L on the circle C, drop a perpendicular from L to the x-axis, and let M be their intersection. Similarly drop a perpendicular from L to the y-axis and call the intersection N. Let P be the point on MN such that LP and MN are perpendicular. Then P is a point of the Astroid, MN is the tangent to the Astroid at P, and LP the normal at P. If D is the intersection of LP and the circle C, and D' is the reflection of D thru MN, then D' is the center of osculating circle at P.



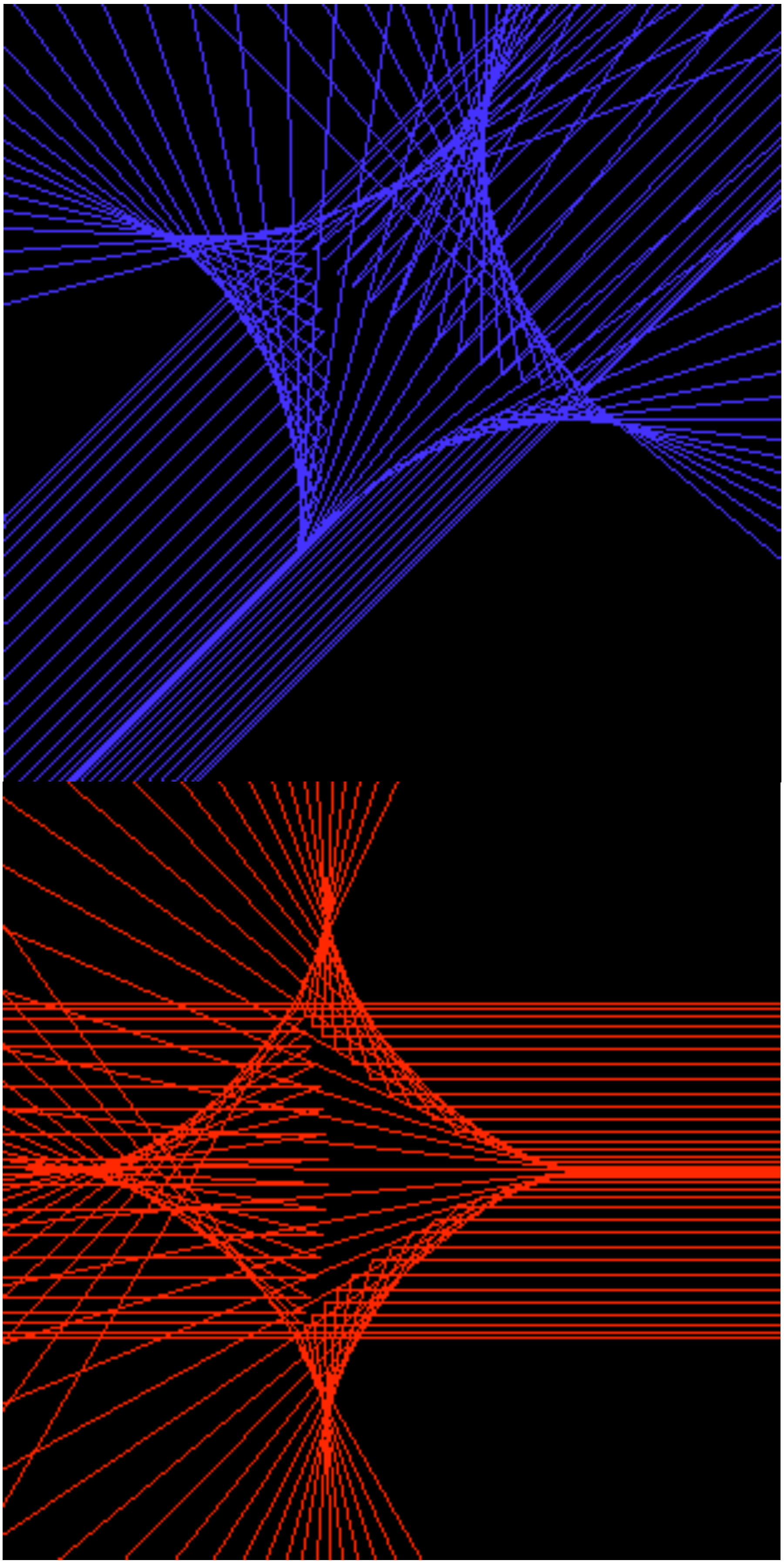
Pedal, Radial, and Rose

The pedal of an Astroid with respect to its center is a 4-petaled rose, called a quadrifolium. The Astroid's radial is also a quadrifolium. (For any epi- or hypo- cycloid, the pedal and radial are equal, and is a rose.)

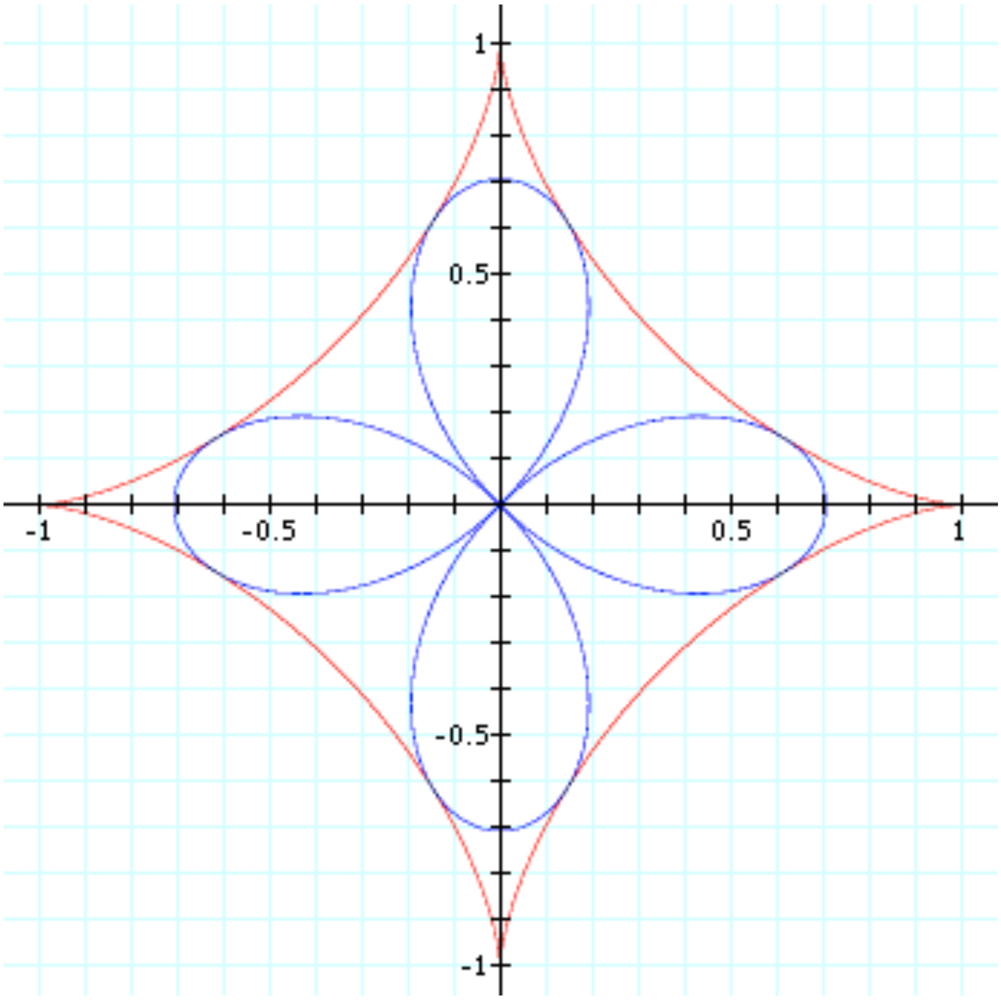
Catacaustic and Deltoid

The catacaustic of a Deltoid with respect to parallel rays in any direction is an Astroid.



Orthoptic

We recall that the *orthoptic* of a curve C is the locus of points P where two tangents to C meet at right angles. The orthoptic of the Astroid is the quadriffolium $r^2 = (1/2)\cos(2\theta)^2$. [Robert C. Yates.]



XL.

Deltoid *

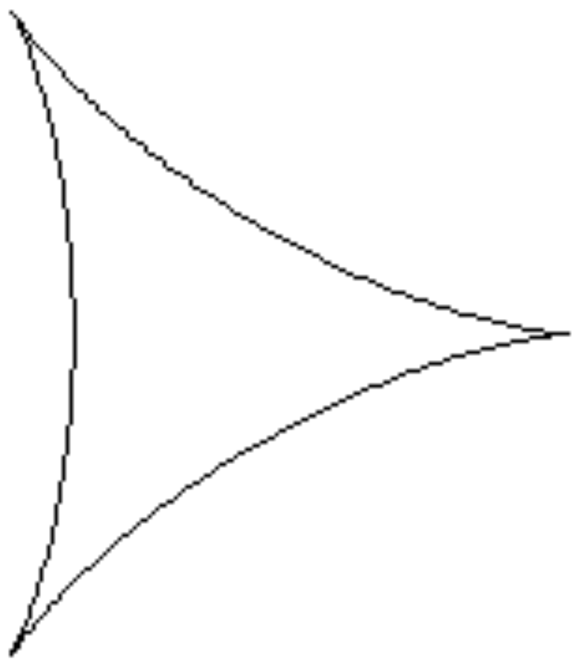
The Deltoid curve was conceived by Euler in 1745 in connection with his study of caustics.

Formulas in 3D-XplorMath:

$x = 2\cos(t) + \cos(2t), \quad y = 2\sin(t) - \sin(2t), \quad 0 < t \leq 2\pi,$

and its implicit equation is:

$(x^2 + y^2)^2 - 8x(x^2 - 3y^2) + 18(x^2 + y^2) - 27 = 0.$

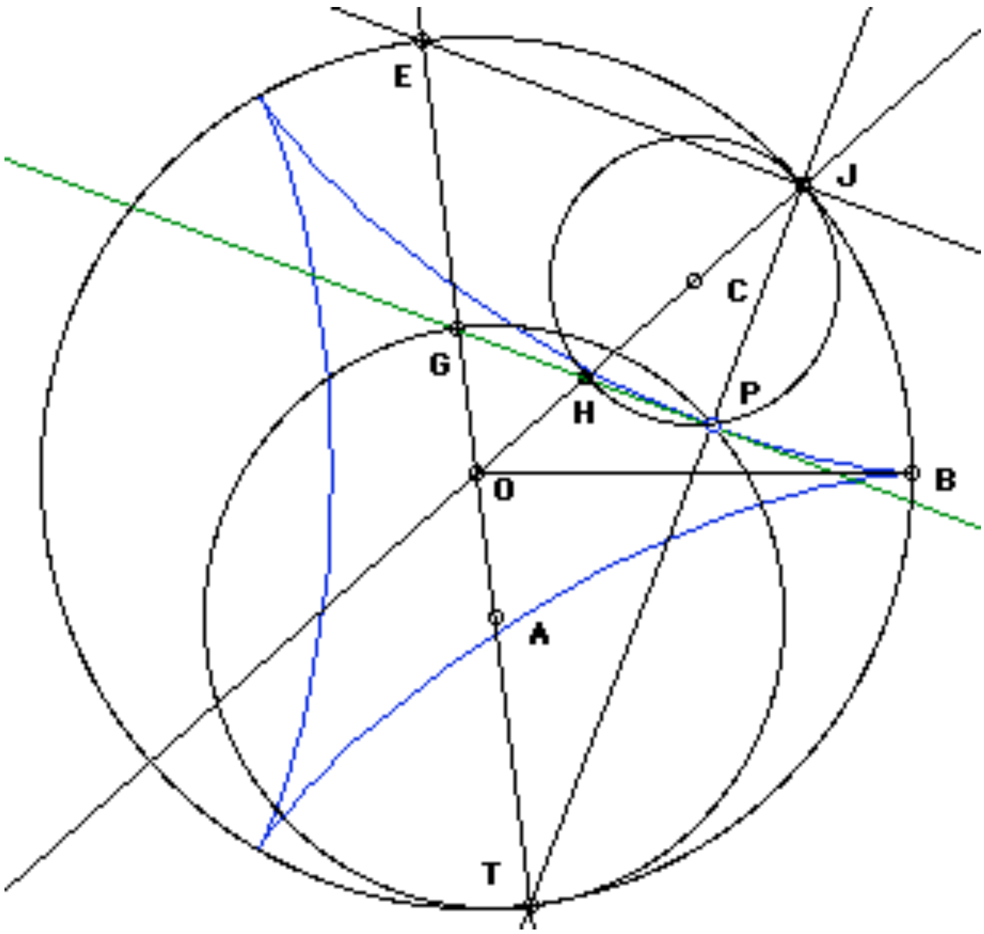


The Deltoid or Tricuspid

The Deltoid is also known as the Tricuspid, and can be defined as the trace of a point on one circle that rolls inside

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

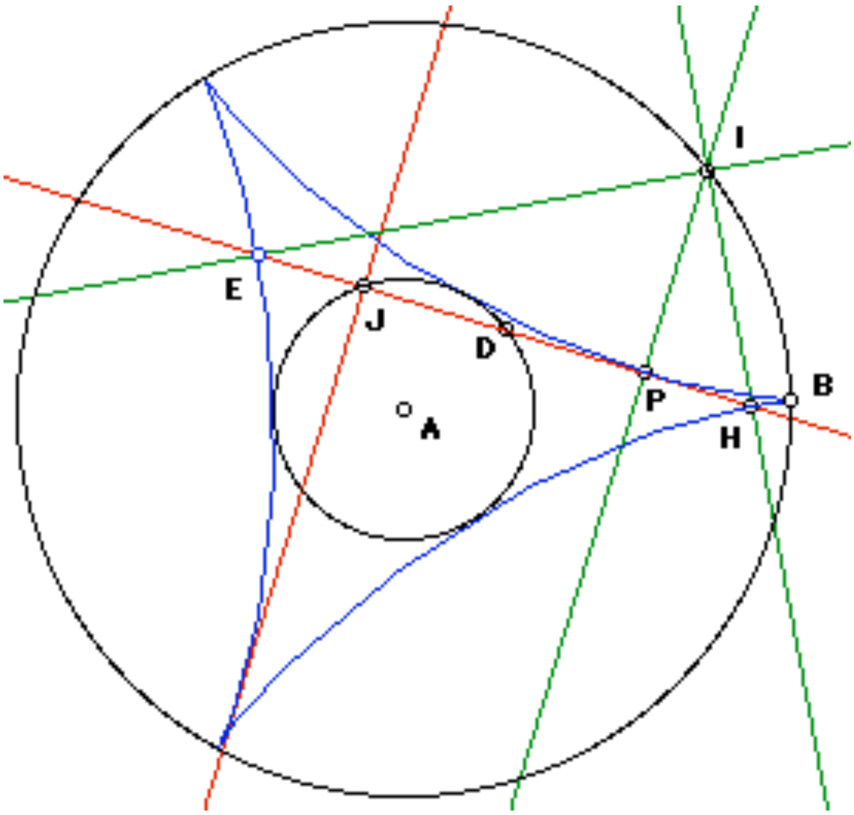
another circle of 3 or 3/2 times as large a radius. The latter is called double generation. The figure below shows both of these methods. O is the center of the fixed circle of radius a, C the center of the rolling circle of radius $a/3$, and P the tracing point. OHCJ, JPT and TAOGE are colinear, where G and A are distant $a/3$ from O, and A is the center of the rolling circle with radius $2a/3$. PHG is colinear and gives the tangent at P. Triangles TEJ, TGP, and JHP are all similar and $TP/JP = 2$. Angle JCP = 3∗Angle BOJ. Let the point Q (not shown) be the intersection of JE and the circle centered on C. Points Q, P are symmetric with respect to point C. The intersection of OQ, PJ forms the center of osculating circle at P.



The Deltoid has numerous interesting properties.

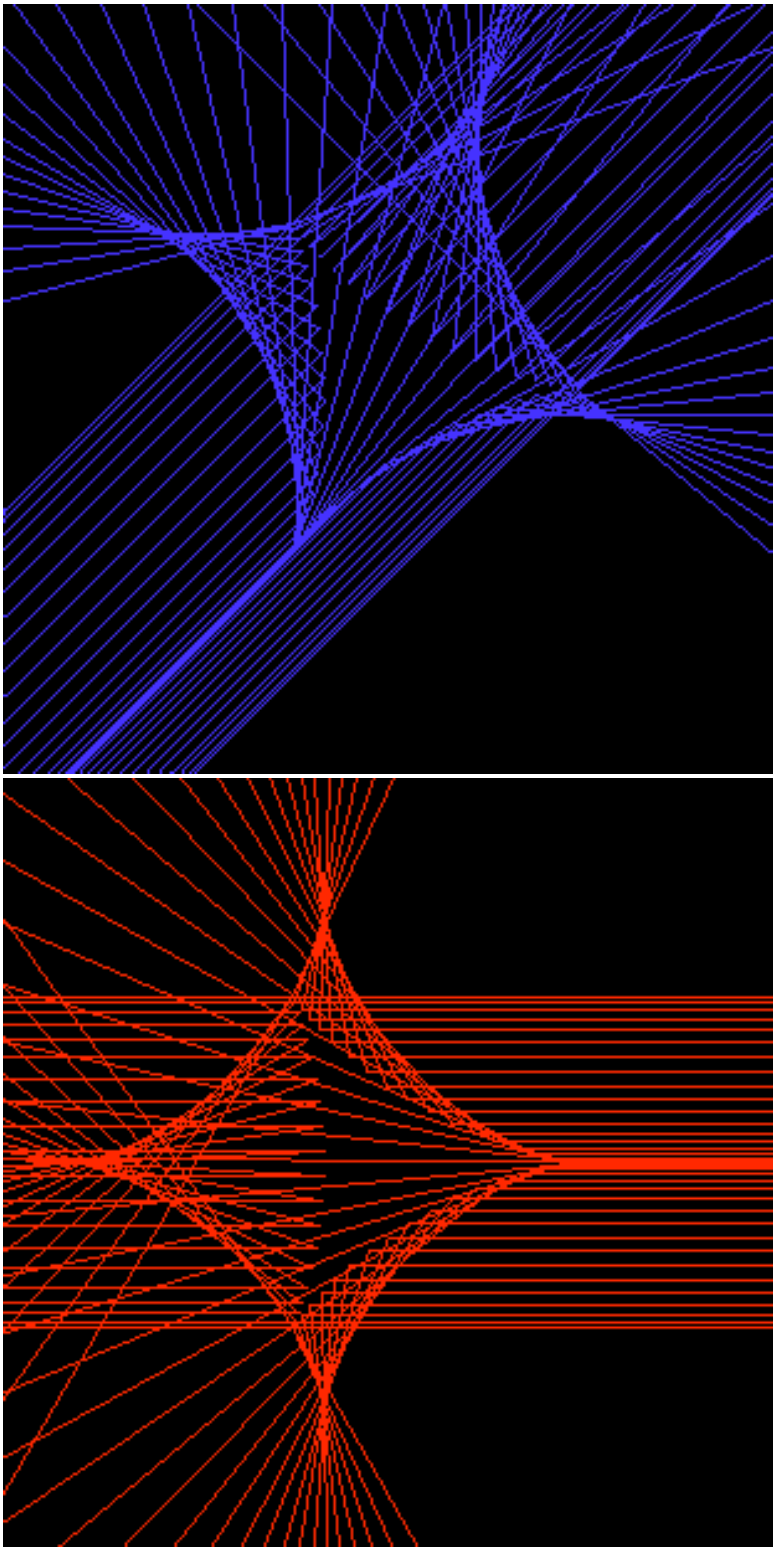
Properties

Tangent



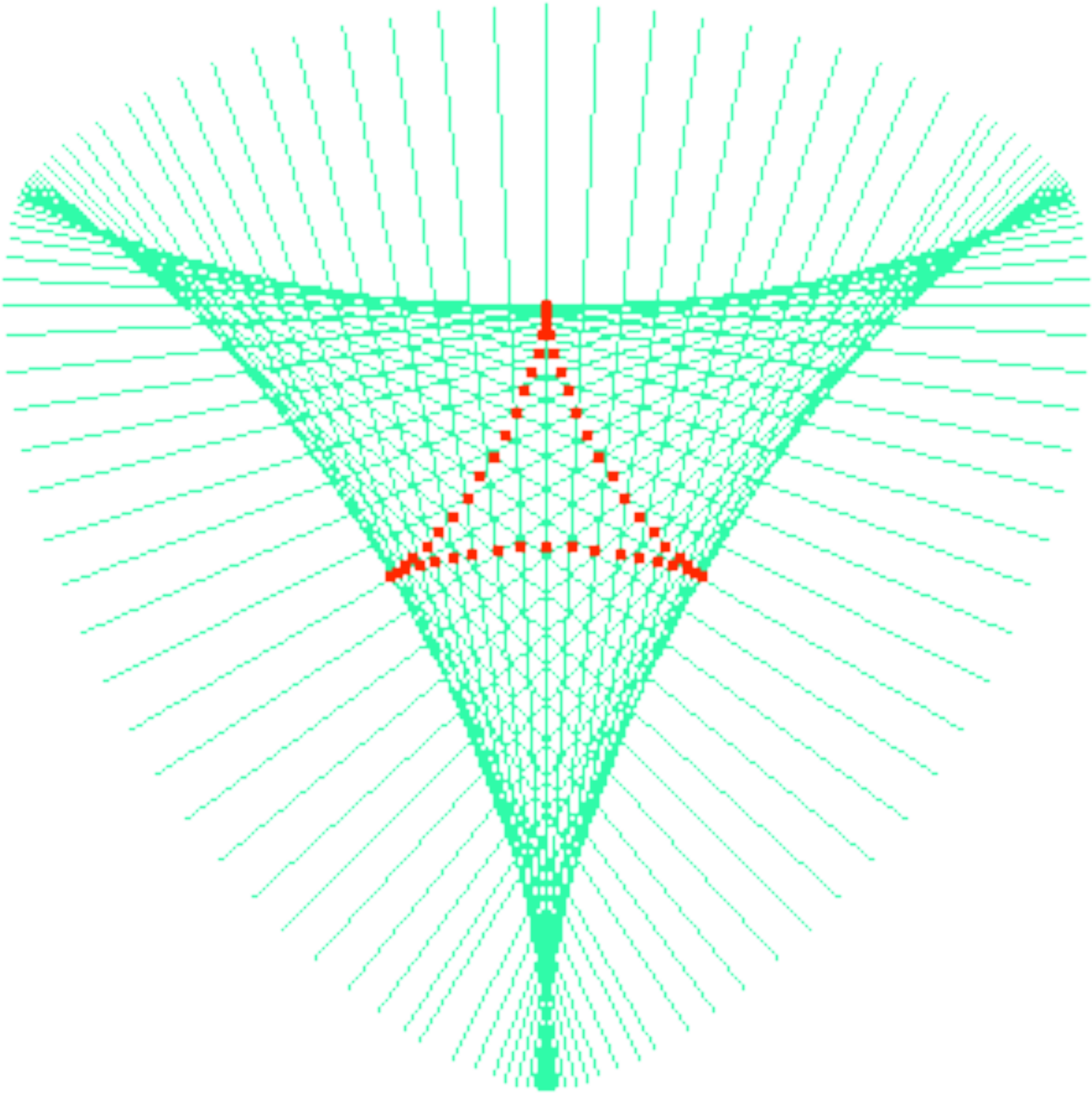
Let A be the center of the curve, B be one of the cusp points,and P be any point on the curve. Let E, H be the intersections of the curve and the tangent at P. The segment EH has constant length distance[E,H]== 4/3*distance[A,B]. The locus of midpoint D of the tangent segment EH is the inscribed circle. The normals at E,P,H are concurrent, and the locus of these intersections is the circumscribed circle. If J is the intersection of another tangent, cutting EH at right angle, then the locus traced by J (the Deltoid's orthoptic) is the inscribed circle.

The Deltoid and the Astroid



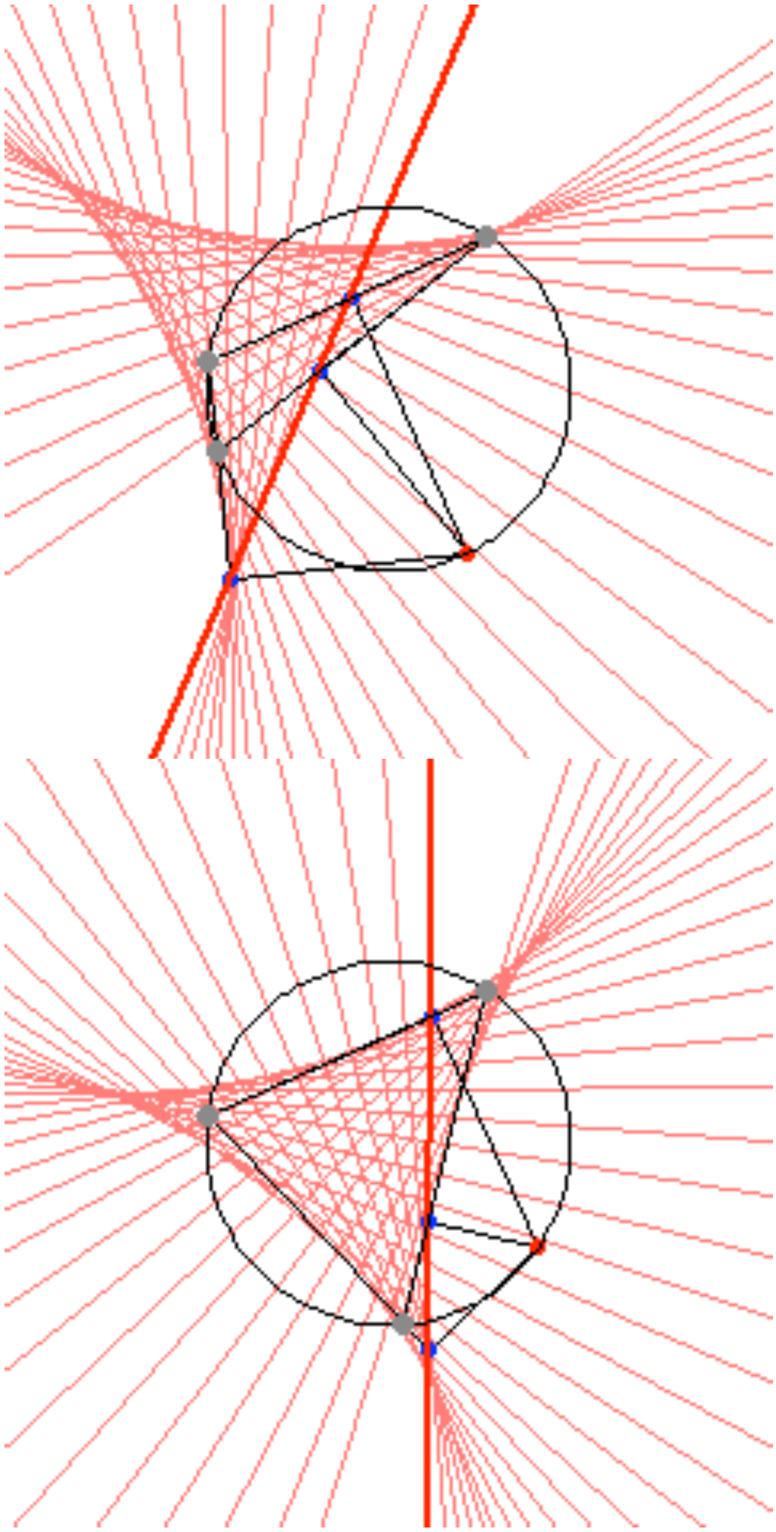
The caustic of the Deltoid with respect to parallel rays in any direction is an Astroid.

Evolute



The evolute of Deltoid is another Deltoid. (In fact, the evolutes of all epicycloids and hypocycloids are scaled version of themselves.) In the above figure, the evolute is shown as the envelope of its normals.

Simson Lines

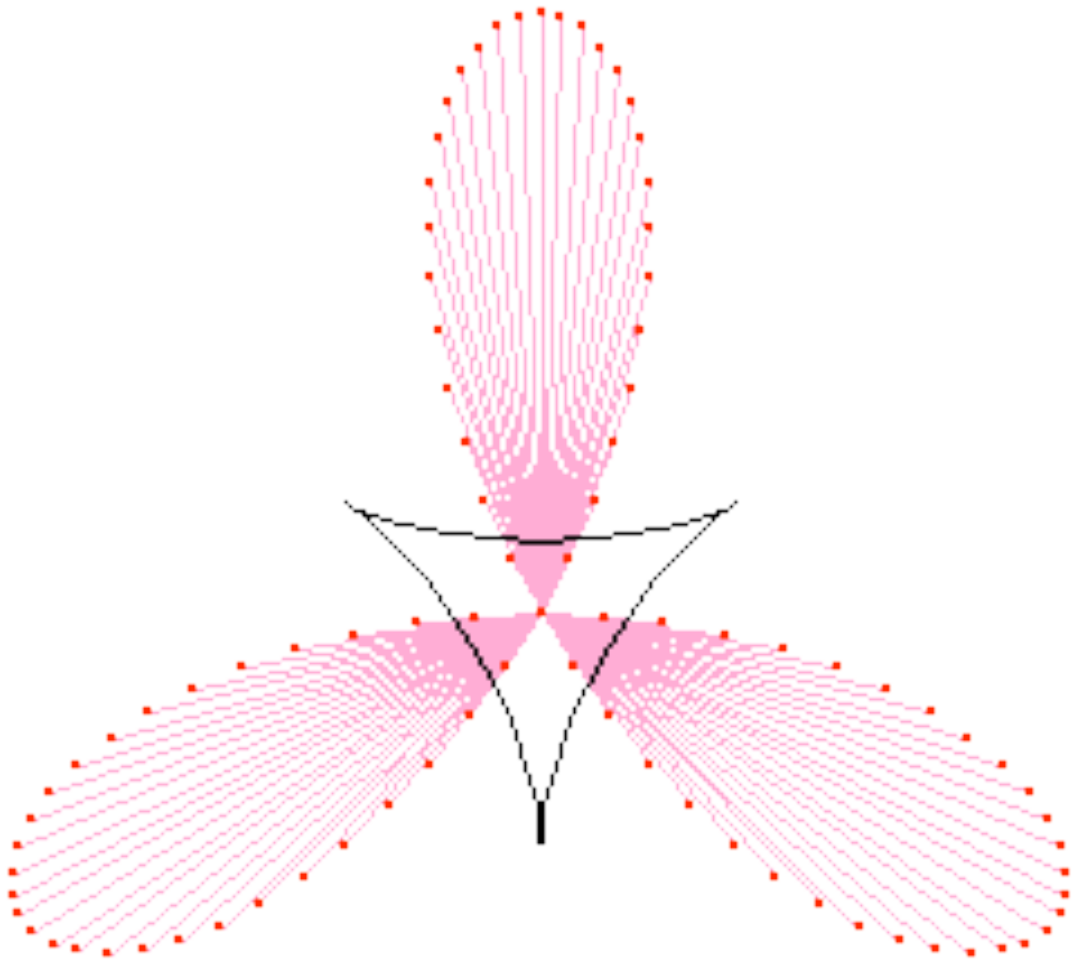


The Deltoid is the envelope of the Simson lines of any triangle. (Robert Simson, 1687–1768)

Step by step description:

1. Let a triangle be inscribed in a circle. 2. Pick any point P on the circle. 3. Mark a point Q1 on any side of the triangle such that line[P,Q1] is perpendicular to it, extending the side if necessary. 4. Similarly, find points Q2 and Q3 with respect to P for the other two sides. 5. The points Q1, Q2, and Q3 are colinear, and the line passing through them is called the Simson line of the triangle with respect to P. 6. Find Simson lines for the other points P on the circle. Their envelope is the deltoid. Amazingly, this is true for any triangle.

Pedal, Radial, and Rose



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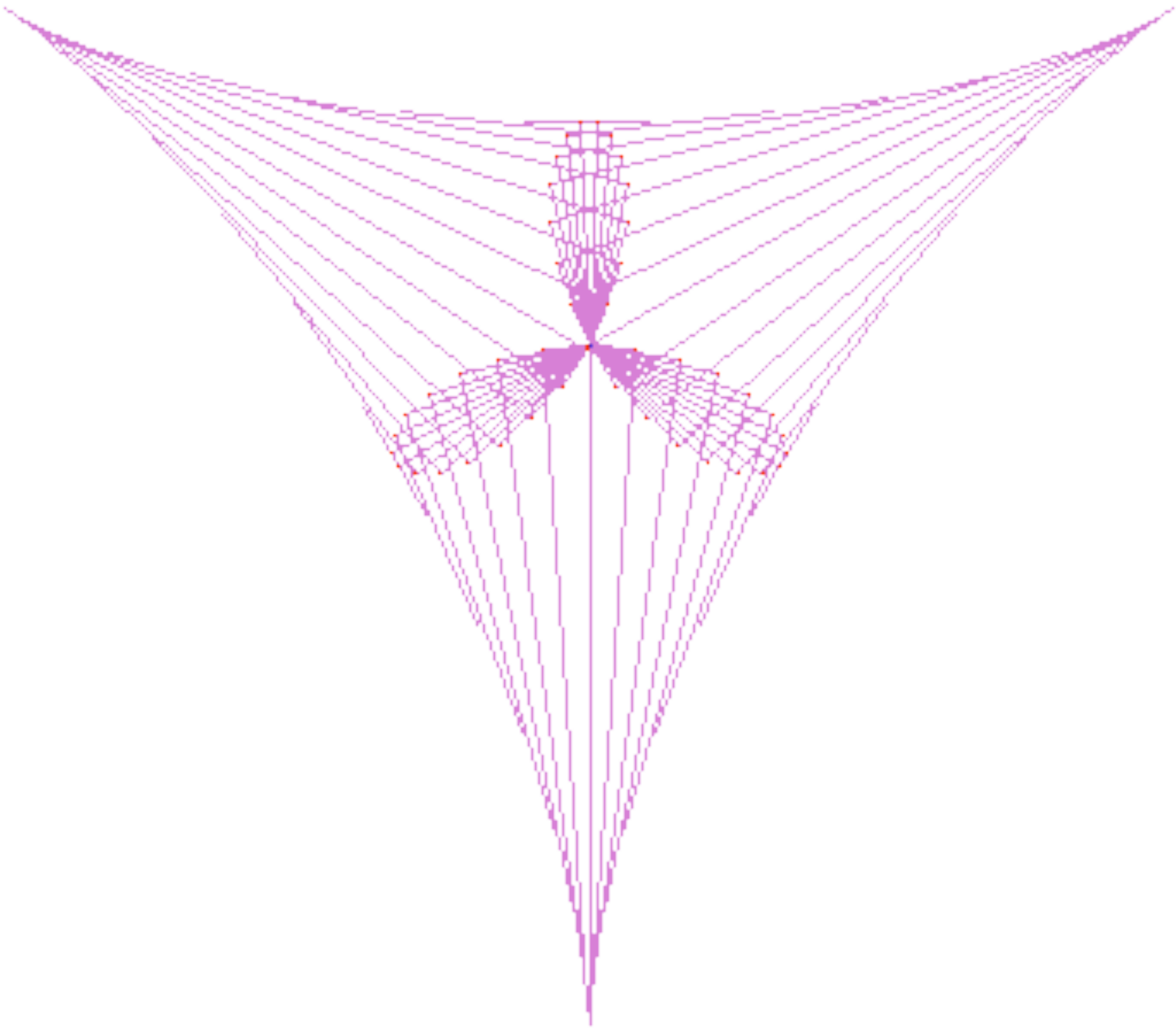
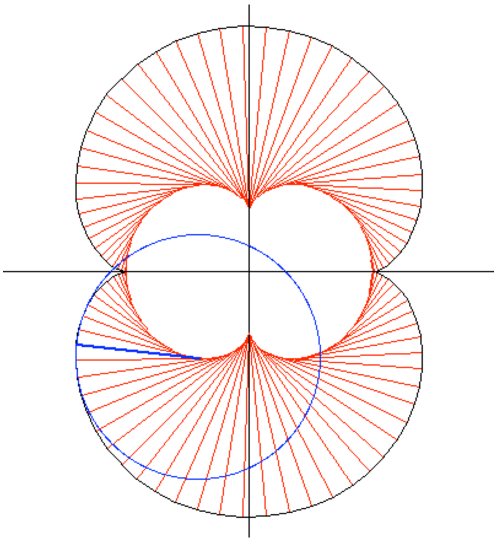


The pedal curve of a Deltoid with respect to a cusp, vertex, or center is a folium curve with one, two, or three loops respectively. The last one is called the trifolium, a three petalled rose. The Deltoid's radial is a trifolium too.

XL.

The Nephroid *

The Nephroid is generated by rolling a circle of one radius on the outside of a second circle of twice the radius. In 3D-XplorMath, either choose `Nephroid` from the Plane Curve menu, or choose `Circle`, then select `Set Parameters...` from the Settings menu and set $hh = -0.5 \cdot aa$, $ii = 1$. With $R = 3r$ we thus have the parametrization for Nephroids:



$$x(t) = R \cdot \sin(t) + r \cdot \sin(3t)$$
$$y(t) = R \cdot \cos(t) + r \cdot \cos(3t)$$

As with Cardioids and Limaçons, one can also make the radius for the drawing stick shorter or longer: Set $ii > 1$ for the looping relatives of the Nephroid or see the default `Morph` in the Animation menu.

The complex map $z \mapsto z^3 + 3z$ maps the unit circle to such

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

a Nephroid. To see this, in the Conformal Map Category, select $z \mapsto z^e e + eez$ from the Conformal Map menu, and then choose Set Parameters from the Settings menu and set ee to 3.

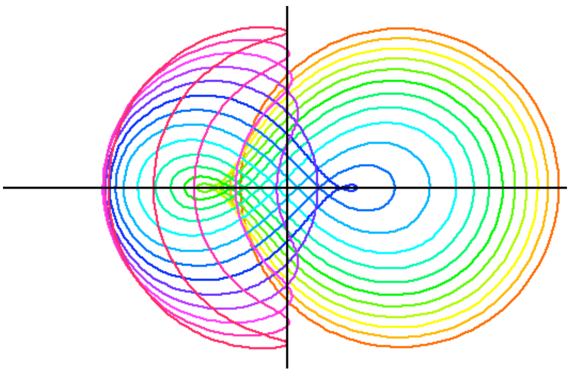
The normals of one Nephroid have as envelope another, smaller Nephroid—the same phenomenon as for the Cardioid and the Cycloid. (To see this select `Show Osculating Circles With Normals` from the Action menu). In technical jargon, the caustics for each of these curves is a similar curve.

H.K.

About Mechanically Generated Curves *

Examples

Presently we have the following mechanically generated plane curves programmed together with a decoration which shows this generation and the corresponding construction of the tangents of the curve:



Epi- and Hypocycloids, all other rolling curves. Also: Tractrix, Cissoid, Conchoid, Lemniscate.

This image is obtained with `Color Morph` in the Animation menu, it shows the family obtained from the current drawing mechanism (here Lemniscate).

Moving Planes

It is often convenient to discuss such mechanical generations in terms of two planes, a fixed plane on which the drawing is done (paper plane) and a second plane which is attached to that piece of the mechanical contraption that holds the drawing pen (drawing plane). In the case of

* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

rolling curves we have the drawing plane attached to the rolling wheel, in the case of the Lemniscate the drawing plane is attached to the middle one of the three connected moving segments.

> *We think of the orbits of the points of the drawing plane as curves that are mechanically generated by the apparatus under consideration.*

The velocity vectors of these orbits clearly give a time dependent vector field. Since this vector field is obtained by differentiating the orbits of a family of **isometries** we obtain at each time t the vector field of a Euclidean **group of motions**, in other words: for most t the vector field consists of the velocity vectors of a rotation, a rotation around the so called momentary center of rotation. This way of looking at the generation gives immediately tangent constructions for all orbits: join the momentary center of rotation to the moving point, the perpendicular line through the point is tangent to its orbit.
It is therefore useful to visualize the movement of the drawing plane together with the time dependent velocity field of its points. We have done this by decorating the drawing plane with not too many but enough random points so that the movement of the drawing plane becomes visible, but the curve under consideration is not obscured. Moreover, to make the vector field visible at each moment t, we have drawn the random points not once, but at two subsequent positions. This picture is interpreted by the brain correctly.

Finally, one has to determine the momentary center of rotation. This is different for each construction. For rolling curves the definition of “rolling” is such that that point, where the rolling wheel touches the fixed curve (“street”), is the momentary center of rotation. In general one has to look for points of the mechanical apparatus for which the direction of the momentary movement (“orbit tangent”) can be decided. The momentary center is then on the line (“radius”) perpendicular to the tangent, so that two such lines are needed. The 3DXM demos use green lines to determine the momentary center.

H.K.

Cubic Curves*

Real Cubic Curves in $\mathbb{R}^2$ were studied extensively by Newton. Later these curves were considered as the real points of complex curves in $\mathbb{C}^2$. If they do not have double points they can be parametrized by additive groups. This means that the points on these curves can be added. Surprisingly this addition is a *geometric addition*, i.e. the sum $P+Q$ can be geometrically constructed from P, Q and the curve. In the case of cubic curves we have:

$$P+Q+R=0 \quad \Leftrightarrow$$

A straight line intersects the curve in P, Q, R .

In 3D-XplorMath are the following examples:

Cubic Polynomial Graph, $x(t)=t, y(t)=x(t)^3+aa\cdot x(t)$,

Cuspidal Cubic, $x(t)=3t^2/(4aa), y(t)=t(t^2+bb)/(4aa^2)$,

Cubic Rational Graph I, $x(t)=\tan(t/2)/aa,\ y(t)=\sin(t)$,

Cubic Rat'l Graph II, $x(t)=\tanh(t/2)/aa,\ y(t)=\sinh(t)$,

Elliptic Cubic, $x+1/x-aa\cdot(y-1/y)=ff$ (implicit),

Folium, $[x,\ y]=aa[(t^2-t^3),\ (t-2t^2+t^3)]/(1-3t-3t^2)$,

Nodal Cubic, $x=1-t^2,\ y=((1-t^2)+bb)\cdot t$.

The last two of these are cubics with one double point.

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

Their points do not form a group, therefore their Action Menu has only the Standard Actions for plane curves.

All the others are parametrized by a 1-dim. Abelian group and the curves are shown with a default demo explaining the geometric addition.
If we intersect a Cubic Polynomial Graph without quadratic term with a line, then the x-coordinates of the intersection points are always roots of a polynomial without quadratic term. In other words: these three x-coordinates add up to 0., the *geometric addition* is the standard addition on the x-axis.
The Cuspidal Cubic is also parametrized by $\mathbb{R}$ (or $\mathbb{C}$) and a simple computation shows: if $1/t_1 + 1/t_2 + 1/t_3 = 0$ then the three points $[x(t_i), y(t_i)]$ lie on a straight line. And dually, if $t_1 + t_2 + t_3 = 0$ then the tangents at the three points $[x(t_i), y(t_i)]$ pass through one point. Again, the addition has a simple geometric interpretation that allows to construct, if two points and the curve are given, their sum.
The first Rational Cubic Graph is parametrized by a circle $\mathbb{S}^1$ (we have to add the infinite point $(\infty, 0)$). The demo that comes with the curve shows how the sum point can be constructed by intersecting lines. The Action Menu offers a second demo that shows how addition on the parametrizing circle and on the curve are the same.
The second Rational Cubic Graph would not be here if we could visualize these curves over the complex Numbers. Over $\mathbb{C}$ one can think of this curve as the group $\mathbb{S}^1 \oplus \mathbb{R}$, a cylinder. The first rational graph visualizes the equator

circle, the second one visualizes the generator through the neutral element plus the opposite generator: two copies of $\mathbb{R}$ (and a double point at ∞).
The Elliptic Cubic is parametrized by a pair of so called *Elliptic Functions.* Such functions can be viewed either as doubly periodic functions from $\mathbb{C}$ to $\mathbb{S}^1$ or as functions defined on some torus. For more details see the text *Elliptic Functions.*
The addition on elliptic curves can be compared with addition on the circle. The formulas for trigonometric functions

$$\begin{aligned}\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)\\ \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\end{aligned}$$

show that

$$(x_1, y_1) \oplus (x_2, y_2) := (x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2)$$

gives the addition of points $(x_1, y_1), (x_2, y_2) \in \mathbb{S}^1$. Notice that the rational points (i.e. the pythagorean triples) are a subgroup. The elliptic functions have analogous functional equations which are similarly the basis for addition formulas for points on elliptic cubics. This is explained in the text *Geometric Addition on Cubic Curves.*

Symmetries Of Elliptic Functions*

[The approach below to elliptic functions follows that given in section 3 of "The Genus One Helicoid and the Minimal Surfaces that led to its Discovery", by David Hoffman, Hermann Karcher, and Fusheng Wei, published in Global Analysis and Modern Mathematics, Publish or Perish Press, 1993. For convenience, the full text of section 3 (without diagrams) has been made an appendix to the chapter on the Conformal Map Category in the documentation of 3D-XplorMath.]

An elliptic function is a doubly periodic meromorphic function, $F(z)$, on the complex plane $\mathbb{C}$. The subgroup $\mathbb{L}$ of $\mathbb{C}$ consisting of the periods of F (the period lattice) is isomorphic to the direct sum of two copies of $\mathbb{Z}$, so that the quotient, $T = \mathbb{C}/\mathbb{L}$, is a torus with a conformal structure, i.e., a Riemann surface of genus one. Since F is well-defined on $\mathbb{C}/\mathbb{L}$, we may equally well consider it as a meromorphic function on the Riemann surface T.

It is well-known that the conformal equivalence class of such a complex torus can be described by a single complex number. If we choose two generators for $\mathbb{L}$ then, without changing the conformal class of $\mathbb{C}/\mathbb{L}$, we can rotate and scale the lattice so that one generator is the complex num-

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* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

ber 1, and the other, τ, then determines the conformal class of T. Moreover, τ_1 and τ_2 determine the same conformal class if and only if they are conjugate under $SL(2, Z)$.

The simplest elliptic functions are those defining a degree two map of T to the Riemann sphere. We will be concerned with four such functions, that we call JD, JE, JF, and WP. The first three are closely related to the classical Jacobi elliptic functions, but have normalizations that are better adapted to certain geometric purposes, and similarly WP is a version of the Weierstrass $\wp$-function, with a geometric normalization. Any of these four functions can be considered as the projection of a branched covering over the Riemann sphere with total space T, and as such it has four branch values, i.e., points of the Riemann sphere where the ramification index is two. For JD there is a complex number D such that these four branch values are $\{D, -D, 1/D, -1/D\}$. Similarly for JE and JF there are complex numbers E and F so that the branch values are $\{E, -E, 1/E, -1/E\}$ and $\{F, -F, 1/F, -1/F\}$ respectively, while for WP there is a complex number P such that the branch values are $\{P, -1/P, 0, \infty\}$. The cross-ratio, λ, of these branch values (in proper order) determines τ and likewise is determined by τ.

The branch values E, F, and P of JE, JF, and WP can be easily computed from the branch value D of JD (and hence from dd) using the following formulas:

$$E = (D - 1)/(D + 1), \quad F = -i(D - i)/(D + i),$$
$$P = i(D^2 + 1)/(D^2 - 1),$$

and we will use D as our preferred parameter for describing the conformal class of T. In 3D-XplorMath, D is related to the parameter dd (of the Set Parameter... dialog) by $D = \exp(dd)$, i.e., if $dd = a + ib$, the $D = \exp(a)\exp(ib)$. This is convenient, since if D lies on the unit circle (i.e., if dd is imaginary) then the torus is rectilinear, while if D has equal real and imaginary parts (i.e., if $b = \pi/4$) then the torus is rhombic. (The square torus being both rectilinear and rhombic, corresponds to $dd = i \cdot \pi/4$).

To completely specify an elliptic function in 3D-XplorMath, choose one of JD, JE, JF, or WP from the Conformal Map menu, and specify dd in the Set Parameter... dialog. (Choosing Elliptic Function from the Conformal map menu will give the default choices of JD and a square torus.)

When elliptic functions where first constructed by Jacobi and by Weierstrass these authors assumed that the lattice of the torus was given. On the other hand, in Algebraic Geometry, tori appeared as elliptic curves. In this representation the branch values of functions on the torus are given with the equation, while an integration of a holomorphic form (unique up to a multiplicative constant) is required to find the lattice. Therefore the relation between the period quotient τ (or rather its $SL(2,Z)$-orbit) and the cross ratio lambda of the four branch values has been well-studied. More recently, in Minimal Surface Theory, it was also more convenient to assume that the branch values of a degree two elliptic functions were given and that the periods had to be computed. Moreover, symmetries became more important than in the earlier studies.

Note that the four branch points of a degree two elliptic function (also called "two-division points", or Zweiteilungspunkte) form a half-period lattice. There are three involutions of the torus which permute these branch points; each of these involutions has again four fixed-points and these are all midpoints between the four branch points. Since each of the involutions permutes the branch points, it transforms the elliptic function by a Moebius transformation. In Minimal Surface Theory, period conditions could be solved without computations if those Moebius transformations were not arbitrary, but rather were isometric rotations of the Riemann sphere—see in the Surface Category the minimal surfaces by Riemann and those named Jd and Je. This suggested the following construction: As degree two MAPS from a torus ($T = \mathbb{C}/\mathbb{L}$) to a sphere, we have the natural quotient maps $T/{-id}$; these maps have four branch points, since the 180 degree rotations have four fixed points. To get well defined FUNCTIONS we have to choose three points and send them to $\{0, 1, \infty\}$. We choose these points from the midpoints between the branch points, and the different choices lead to different functions. The symmetries also determine the points that are sent to $\{-1, +i, -i\}$. In this way we get the most symmetric elliptic functions, and they are denoted JD, JE, JF. The program allows one to compare them with Jacobi's elliptic functions. The function $WP = JE * JF$ has a double zero, a double pole and the values $\{+i, -i\}$ on certain midpoints (diagonal ones in the case of rectangular tori). Up to an additive and a multiplicative constant it agrees with the Weierstrass $\wp$-function, but in our normalization it is the Gauss map of Riemann's minimal surface on each rectangular torus.

We compute the J-functions as follows. If one branch value is called +B, then the others are $\{-B, +1/B, -1/B\}$. Therefore the function satisfies the *differential equations*

$$(J')^2 = (J'(0))^2(J^4 + 1 - (B^2 + 1/B^2)J^2) = F(J),$$
$$J'' = (J'(0))^2(2J^3 - (B^2 + 1/B^2)J = F'(J)/2).$$

Numerically we solve this with a fourth order scheme that has the analytic continuation of the square root $J' = \sqrt{J'^2}$ built into it:
Let $J(0), J'(0)$ be given. Compute $J''(0) := F'(J(0))/2$ and, for small z,

$$J_m := J(0) + J'(0) \cdot z/2 + J''(0) \cdot z^2/8,\ \ J''_m := F'(J_m)/2,$$
$$J(z) := J(0) + J'(0) \cdot z + (J''(0) + 2 \cdot J''_m) \cdot z^2/6.$$

Finally let $J'(z)$ be that square root of $F(J(z))$ that is closer to $J'(0) + J''_m \cdot z$ (analytic continuation!). Repeat.

H.K.

Addition on Cubic Curves.*

See also the Action Menu of the Parabola “Show Normals through Mouse Point” and the comments in the ATO.
As an introductory example view the unit circle as a group. Then the addition of angles $\phi \in (\mathbb{R} \bmod 2\pi)$ gets translated via the parametrization

$$x = \cos(\phi), y = \sin(\phi)$$

into

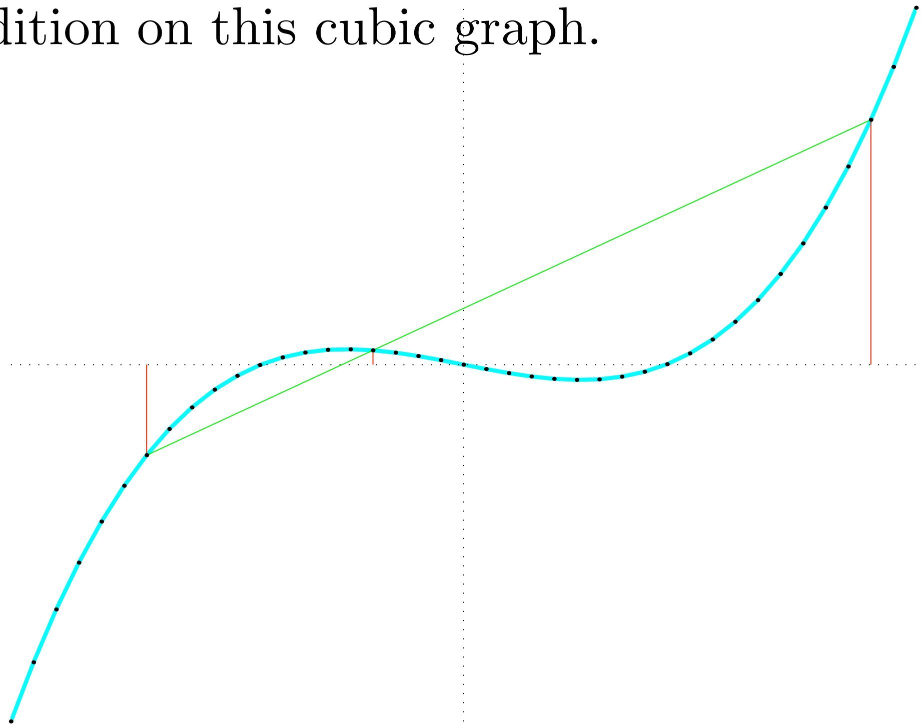
$$(x_1, y_1) \oplus (x_2, y_2) := (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1).$$

Once this addition law is known one does not need the transcendental functions sin and cos to “add” points on the circle. Even to do this addition with ruler and compass is easy. And it is amusing to note that the Pythagorean (or rational) points of the circle are a subgroup, e.g. $(3/5, 4/5) \oplus (12/13, 5/13) = ((36-20)/65, (15+48)/65)$.

In a similar way there exists a geometric addition on cubic curves, and if the cubic is parametrized with appropriate functions (defined either on $\mathbb{C}$, or on $\mathbb{C}/2\pi\mathbb{Z}$, or on $\mathbb{C}/\Gamma$, Γ a lattice in $\mathbb{C}$) then the well known addition in the domain is, under the special parametrization, the same as the geometric addition on the cubic. The simplest instance is when the cubic is the graph of a cubic polynomial without

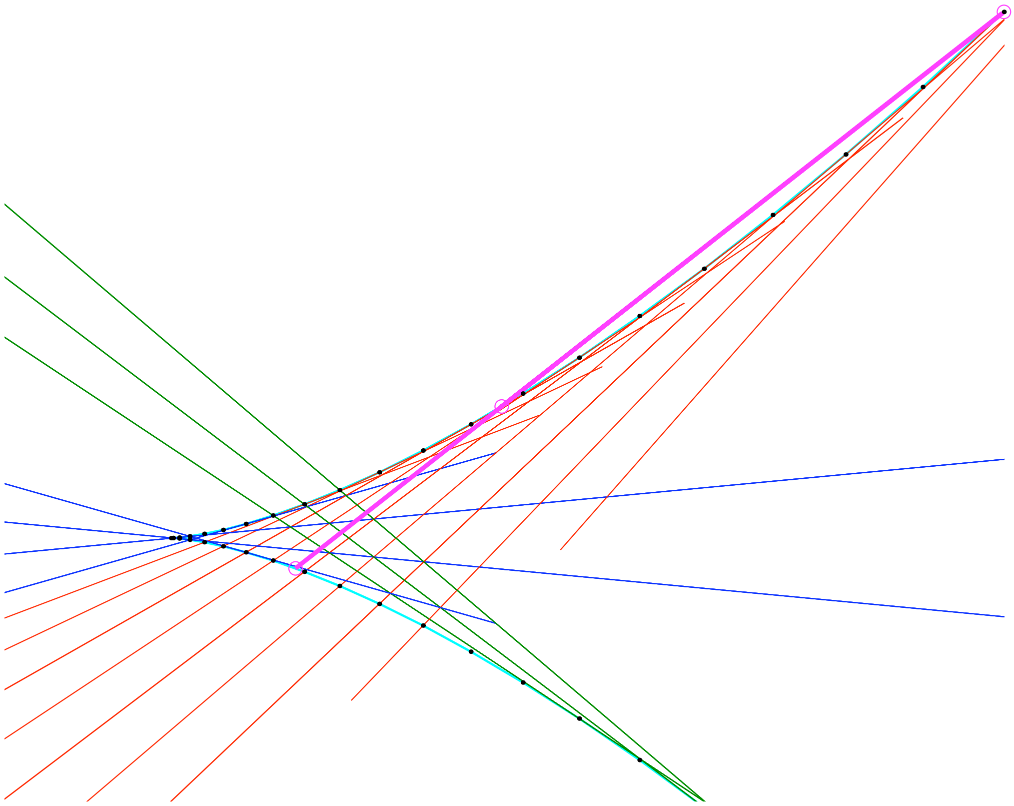
* This file is from the 3D-XplorMath project. Please see: http://3D-XplorMath.org/

quadratic term: $y = x^3 + mz + c$. Then, if we have two points $(x_1, y_1), (x_2, y_2)$ on this cubic and join them by a line, this line intersects the graph in a third point (x_3, y_3) such that $x_1+x_2+x_3 = 0$. This gives a geometric definition of addition on this cubic graph.



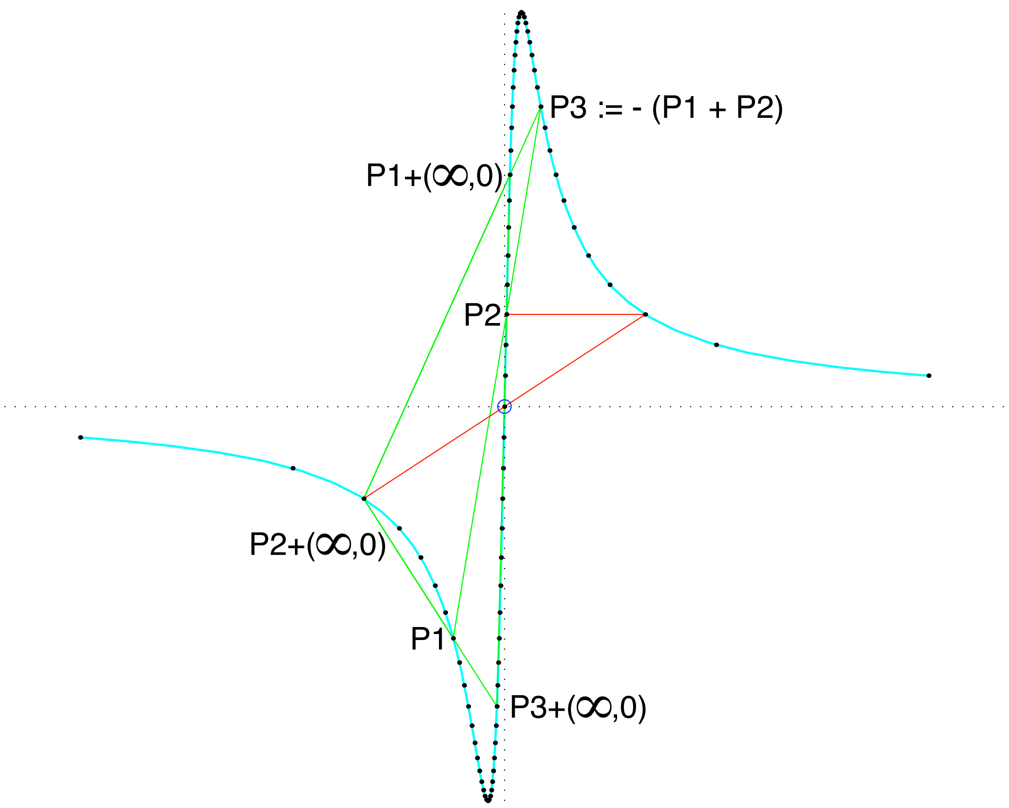
Addition on a polynomial cubic graph without quadratic term. Every line intersects so that $x_1 + x_2 + x_3 = 0$. Note discrete subgroup.

Similarly, let us map $\mathbb{C}$ bijectively onto the Cuspidal Cubic by $z \mapsto (z^2, z^3)$. In this case, if we have $z_1 + z_2 + z_3 = 0$, then the tangents at the three points (z_j^2, z_j^3) are concurrent—we have seen this as a property of the Parabola, because the Cuspidal Cubic is the evolute of the Parabola. One can also see the previous colinearity as reflecting addition, because the three points $(z_j^2, z_j^3), j = 1, 2, 3$, of this cubic lie on a line if $1/z_1 + 1/z_2 + 1/z_3 = 0$.



Addition on the cuspidal cubic $z \mapsto (z^2, z^3)$. Note the discrete subgroup. If $z_1 + z_2 + z_3 = 0$, then the tangents at these three points are concurrent. If $1/z_1 + 1/z_2 + 1/z_3 = 0$, then these three points lie on a straight line.

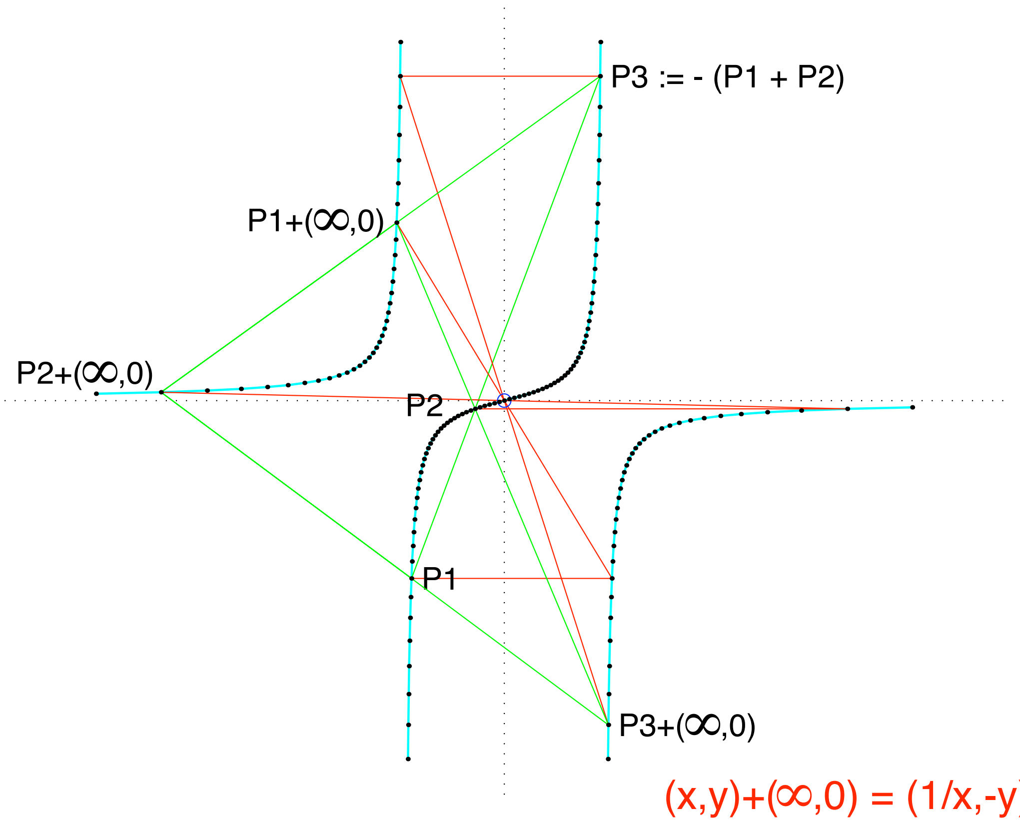
The next case is the group $\mathbb{C}/2\pi\mathbb{Z}$. The trigonometric functions identify points in $\mathbb{C}$ mod 2π. We map this group to a cubic curve by $x := \tan(z/2)$, $y := \sin(z)$, so that $y = 2x/(x^2 + 1)$ and this cubic is again a graph. The addition theorems $\tan(z + w) = (\tan(z) + \tan(w))/(1 - \tan(z)\tan(w))$ and $\sin(z+w) = \sin(z)\cos(w)+\cos(z)\sin(w)$ with $\cos(z) = 1 - 2\sin(z/2)^2 = 1 - \sin(z) \cdot \tan(z/2)$ again give an addition on this cubic graph: it is a geometric addition because the three points (x_j, y_j) lie on one line iff $z_1 + z_2 + z_3 = 0$. The name "geometric addition" is even more justified because the third point (x_3, y_3) can be constructed with ruler and compass from the other two. In fact, for repeated additions a ruler suffices: As a preparation we have to add to all points in sight the 2-division point $(\infty, 0) = (\tan(\pi/2), \sin(\pi))$ as follows: $(x, y) \oplus (\infty, 0) = (-1/x, -y)$. One needs ruler and unit circle for this. Then the lines through $(x_1, y_1), (x_2, y_2)$ and $(x_1, y_1) \oplus (\infty, 0), (x_2, y_2) \oplus (\infty, 0)$ intersect in the point $(x_3, y_3) = -(x_1, y_1) \oplus (x_2, y_2)$.



Addition group $\mathbb{S}^1$ on a cubic that is the graph of $x \mapsto y = 2x/(x^2+1)$, parametrized by $x := \tan(z/2)$, $y := \sin(z)$. Note the finite discrete subgroup. $(\infty, 0) = (\tan(\pi/2), \sin(\pi))$, the point at infinity, is the only point of order 2.

So far we have seen the circle part of the cylinder group $\mathbb{C}/2\pi\mathbb{Z}$. To see a generator of the cylinder we replace t, x, y

by it, ix, iy, then we obtain $x := \tanh(z/2),\ y := \sinh(z)$, so that $y = 2x/(1-x^2)$. The component of the graph through 0 is a subgroup isomorphic to $\mathbb{R}$. It represents one generator of the cylinder. The other two components represent the opposite generator with one point missing: the 2-division point opposite 0 is the point $(\infty, 0)$ on this cubic. This allows the same ruler construction of addition as before, except for a sign change in $(x,y) \oplus (\infty, 0) = (+1/x, -y)$ (because $1/i = -i$).



Addition group $\mathbb{R}\cup\mathbb{R}$ on a cubic that is the graph of $x \mapsto y = 2x/(1-x^2)$ and is parametrized by $x := \tanh(z/2),\ y := \sinh(z)$. $(\infty, 0)$ is the only point of finite order. Note the infinite discrete subgroup with one finite subgroup of order 2.

Finally we come to the group $\mathbb{C}/\Gamma$. The parametrizing functions of the previous example, $\tan(z/2)$, $\sin(z)$, must be replaced by Γ-invariant, "doubly periodic" functions, also called elliptic functions. The simplest of these are those of degree two, as maps from the torus $T^2 := \mathbb{C}/\Gamma$ to the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$. Two facts are important:

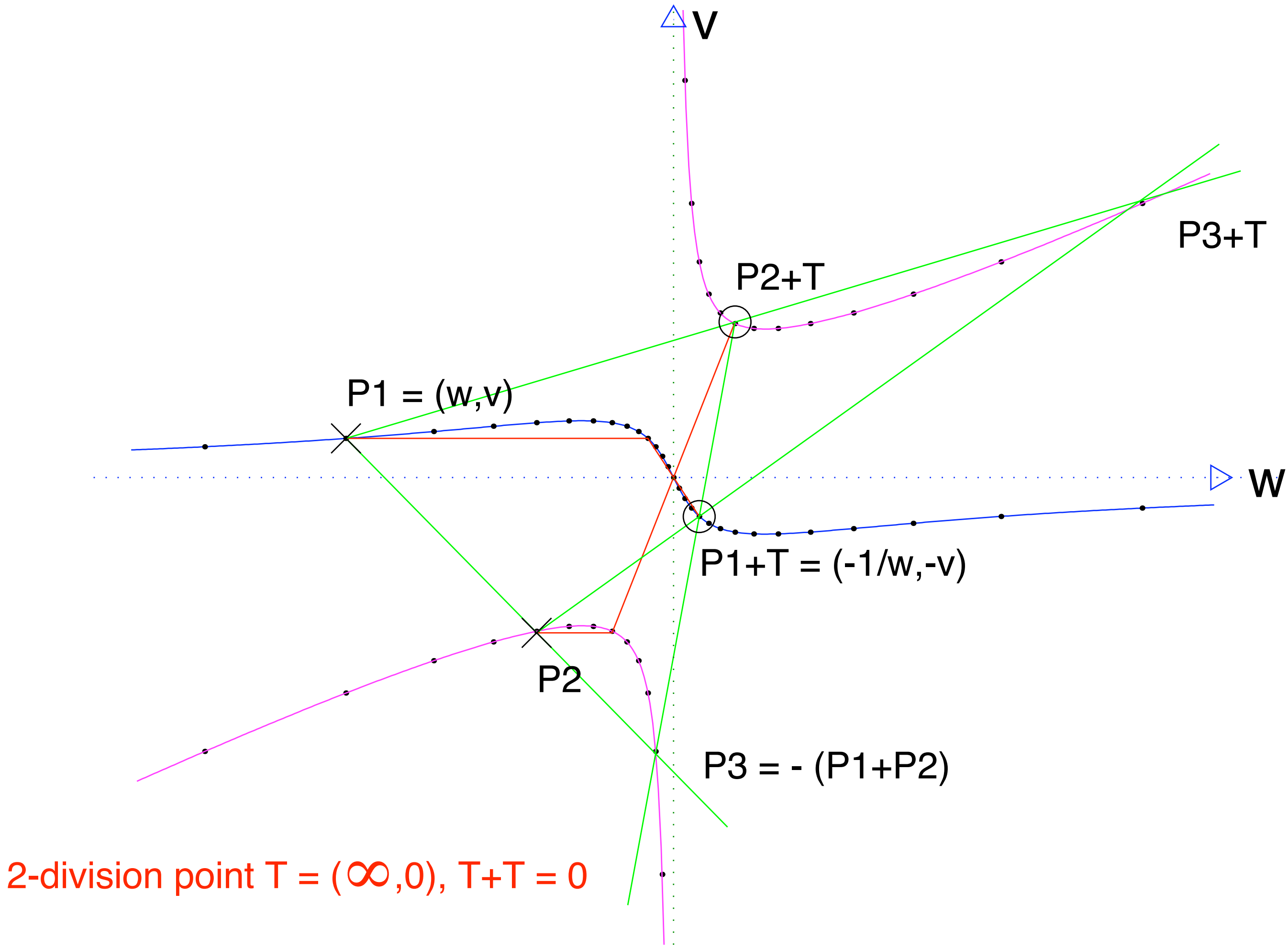
(i) Pairs of such functions satisfy cubic equations such as $(w^2+1)v = \text{const} \cdot (v^2-1)w$. The solution set of any cubic equation is called a cubic curve.

(ii) There are addition formulas, analogous to those for sin and cos.
They determine the pair $(v(z_1+z_2), w(z_1+z_2))$ from the pairs $(v(z_1), w(z_1))$ and $(v(z_2), w(z_2))$.

It turns out that these addition formulas are again "geometric" as in the previous cases, namely, the three pairs $(v(z_1), w(z_1))$, $(v(z_2), w(z_2))$, $(-v(z_1+z_2), -w(z_1+z_2))$ lie on a line. Therefore we can again define addition on the cubic geometrically:

> *Join the points to be added by a line and take the third point of intersection with the cubic as the negative of the sum.*

The addition formulas are simple enough so that the geometric addition is again a "ruler and compass construction". The compass is only needed to add 2-division points as in the previous case, all further additions can be done by intersecting lines only.



(Notice the discrete subgroup)

Addition on a general cubic:

$$(w_1, v_1) \ominus (w_2, v_2) = \left(\frac{w_1 + w_2}{1 - w_1 w_2} \cdot \frac{v_1 - v_2}{v_1 + v_2}, \frac{1 + w_1 w_2}{1 - w_1 w_2} \cdot \frac{v_1 - v_2}{1 - v_1 v_2}\right)$$

.

The elliptic functions v, w, parametrizing the above cubic curve have numerous properties that can be used to define them. For example, they are numerically accessible, since they are solutions of the following system of differential equations (compare $\tan' / \tan = 1/\cot + \cot$):

$$\frac{v'}{v} = w'(0)\left(\frac{1}{w} - w\right),$$
$$\frac{w'}{w} = v'(0)\left(\frac{1}{v} + v\right),$$

with $v'(0)/w'(0) = -2$ for the above cubic. These imply functional equations for v, w so that more similarities with the trigonometric case, like $(\sin')^2 = 1 - \sin^2$, become apparent:

$$\begin{aligned}
\left(\frac{v'}{v}\right)^2 &= w'(0)^2\left(\frac{1}{w} - w\right)^2 \\
&= w'(0)^2\left(\left(\frac{1}{w} + w\right)^2 - 4\right) \\
&= v'(0)^2\left(\left(\frac{1}{v} - v\right)^2\right) - 4w'(0)^2,
\end{aligned}$$

and hence: $$(v')^2 = v'(0)^2\left((1 - v^2)^2 - 4\frac{w'(0)^2}{v'(0)^2} \cdot v^2\right).$$

Every differential equation

$$(f')^2 = F(f) \text{ implies } 2f'' = F'(f).$$

The first order equation determines f' only up to sign while the second order equation determines f'' uniquely, in particular for trigonometric and elliptic functions.

Folium of Descartes*

This is a famous curve with a long history (see e.g. `http:\\www-history.mcs.st-andrews.ac.uk /Curves/Curves.html`). The curve is the solution set of the equation

$$x^3 + y^3 = 3axy.$$

One can see that the solutions for different a differ only by scaling, namely divide the equation by a^3 and replace $x/a,\ y/a$ by x, y.

The two most frequently given parametrizations are:

$$x(t) = \frac{3t}{1+t^3},\ y(t) = \frac{3t^2}{1+t^3},$$
$$r(\varphi) = \frac{\sin 2\varphi}{\sin^3\varphi + \cos^3\varphi},\quad -\pi/4 < \varphi < 3\pi/4.$$

The first parametrization has the disadvantage that at $t = -1$ the denominator vanishes, the curve jumps "from minus infinity to plus infinity", while the important double point at $0 \in \mathbb{R}^2$ is left out (or given by $t = \infty$). This can be remedied by the transformation $u = 1/(1+t),\ t =$

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$-1+1/u$, which changes the parametrization to

$$x(u) = \frac{u^2 - u^3}{1 - 3u + 3u^2}, \; y(u) = \frac{u - 2u^2 + u^3}{1 - 3u + 3u^2},$$
$$-\infty < u < \infty.$$

H.K.

About Implicit Curves in the Plane*

Compare Implicit Surfaces in Space

There are three principal methods for describing curves in the plane:

a) As parametrized curves $c(t) = (x(t), y(t))$ with $x, y : (t_0, t_1) \mapsto \mathbb{R}$. For example the unit circle can be given as

$$x(t) = R \cdot \cos(t), \quad y(t) = R \cdot \sin(t), \quad t \in [0, 2\pi].$$

b) As the graph $y = F(x)$ of a function $F : [x_0, x_1] \mapsto \mathbb{R}$. For example the upper unit semi-circle can be given as the graph of the function $F(x) = \sqrt{1 - x^2}$ for $x \in (-1, +1)$.

c) Implicitly as a level set $\{f = c\}$ of a function $f : \mathbb{R}^2 \mapsto \mathbb{R}$. For example the unit circle is the level $\{f = 1\}$ of the function $f(x, y) = x^2 + y^2$.

Implicit Curves in 3DXM:

Cassini Ovals $f(x, y) = ((x-aa)^2+y^2)((x+aa)^2+y^2)-bb^4$
Tacnodal Quartic $f(x, y) = y^3 + y^2 - x^4$
Teissier singular Sextic $f(x, y) = (y^2 - x^3)^2 - x^5 \cdot y$
Userdefined Implicit Curves: *available*
Parametrized Curves with Level Functions:
Cuspidal Cubic $f(x, y) = 27aa \cdot y^2 - 4(x + bb)x^2$
Nodal Cubic $f(x, y) = y^2 - (1 - x)x^2$

Clearly, method b) can easily be written as a special case

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of methods a) or c) by using for a) the "graph parametrization" $x(t) = t,\ y(t) = F(t)$, and by using for c) the trivial level function $f(x, y) = y - F(x)$ and $\{f = 0\}$.

However, implicit curves $\{f(x, y) = c\}$ really give a different and somewhat richer class of objects than are given by explicit parametrization. For example, level sets may have several components; also, one is more interested in the singularities of level sets. In differential geometry one usually assumes that parametrized curves are without singularities, while in algebraic geometry the singularities of the level sets of polynomials are a major subfield of interest. The *Tacnodal Quartic* and the *Teissier Sextic* are examples in 3DXM.

Up to release 10.6 there are only a small number of implicit curves preprogrammed into 3DXM. What actually gets drawn are the solutions of the equation $f(x, y) = ff$ with $x_{min} \leq x \leq x_{max}, y_{min} \leq y \leq y_{max}$ (where these limits can be set in the *Settings Menu*, dialog entry *Set t,u,v,ranges...* and, as always, *ff* can be set in the dialog entry *Set Parameters, Modify Object.*
Note that user-defined implicit curves can be entered.

The default morphs of the implicit curves vary the parameter *ff*, so that, what one sees is a family of level curves of f. It may be helpful to think of the function $f(x, y)$ as giving the "height above sea-level" at the point (x, y), in which case the levels $\{f(x, y) = ff\}$ are just the level lines one is used to from topographic maps. If one chooses from

the *Animation Menu* the entry *Color Morph*, the program will draw such a topographic map with each level line a different color.
Some parametrized curves are provided with level functions. For these the *Animation Menu* has the entry *Morph Level Lines*. In the Cassini case this morph looks better with morphing parameter $bb = ff^{1/4}$.

Note that, while a parametrized curve depends on parameters only if the author chooses to embed it in some family, implicit curves always come naturally as 1-parameter family of curves. These families have been used to study singularities of curves via limits of nonsingular curves.

Tangents, Normals and Curvature

The gradient of the (height) function f is a vectorfield along and normal to the level lines. Therefore we have, even without parametrizing the curve, normals and tangents:

$$n = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}, \quad t = (-n_y, n_x).$$

Assuming we had a parametrized curve with unit normal and tangent fields n, t then the formula $\dot{n}(s) = \kappa(s) \cdot \dot{c}(s)$ holds whether or not s is arc length parameter. This implies for our vector fields

$$\kappa = \langle \nabla_t n,\ t\rangle = \mathrm{hesse} f(t,t)/|\operatorname{grad} f|.$$

R.S.P.

Cassinian Ovals*

Level function in 3DXM:

$$f(x, y) := (x - aa)^2 + y^2) \cdot ((x + aa)^2 + y^2) - bb^4$$

The default *Color Morph* varies $bb = ff^{1/4}$ instead of ff.

The Cassinian Ovals (or Ovals of Cassini) were first studied in 1680 by Giovanni Domenico Cassini (1625–1712, aka Jean-Dominique Cassini) as a model for the orbit of the Sun around the Earth.

A Cassinian Oval is a plane curve that is the locus of all points P such that the *product of the distances* of P from two fixed points F_1, F_2 has some constant value c, or $\overline{P\,F_1}\,\overline{P\,F_2} = c$. Note the analogy with the definition of an ellipse (where product is replaced by sum). As with the ellipse, the two points F_1 and F_2 are called *foci* of the oval. If the origin of our coordinates is the midpoint of the two foci and the x-axis the line joining them, then the foci will have the coordinates $(a, 0)$ and $(-a, 0)$. Following convention, $b := \sqrt{c}$. Then the condition for a point $P = (x, y)$ to lie on the oval becomes: $((x-a)^2+y^2)^{1/2}((x+a)^2+y^2)^{1/2} = b^2$. Squaring both sides gives the following *quartic polynomial equation* for the Cassinian Oval:

$$((x - a)^2 + y^2)((x + a)^2 + y^2) = b^4.$$

When b is less that half the distance $2a$ between the foci, i.e., $b/a < 1$, there are two branches of the curve. When

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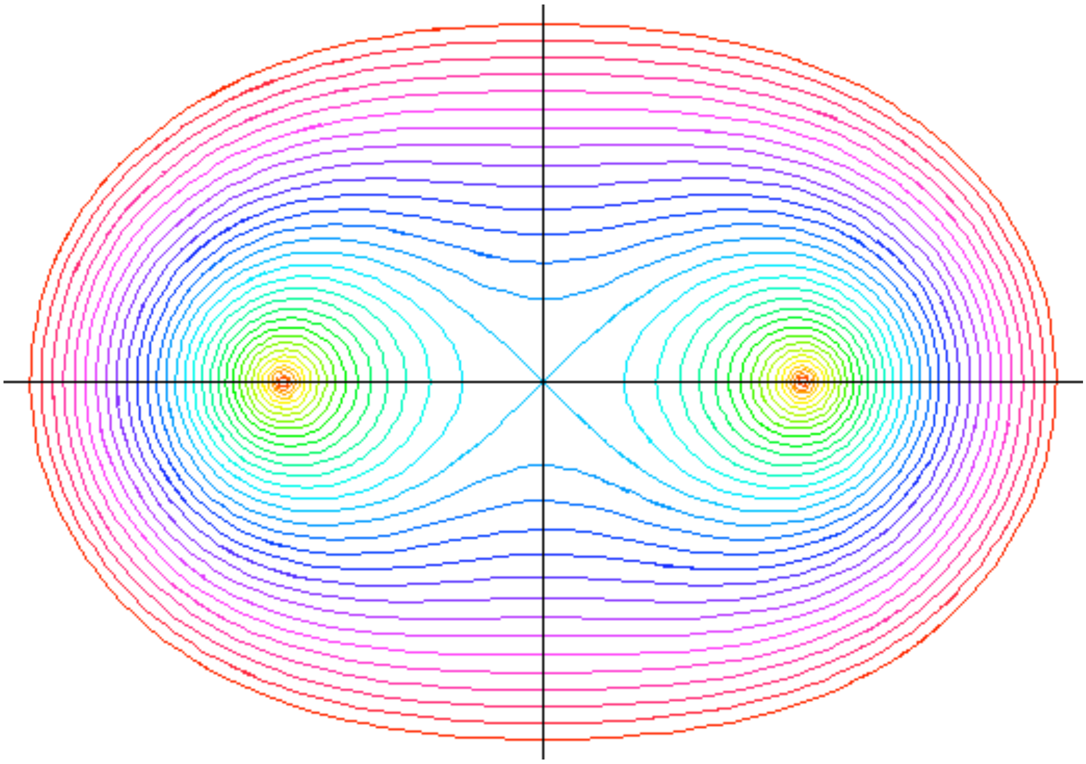
$a = b$, the curve has the shape of a figure eight and is known as the *Lemniscate of Bernoulli*.

The following image shows a family of Cassinian Ovals with $a = 1$ and several different values of b.



In 3D-XplorMath, you can change the value of parameter $b = bb$ in the Settings Menu → SetParameters. An animation of varying values of b can be seen from the Animate Menu → Color Morph.

Bipolar equation: $r_1 r_2 = b^2$

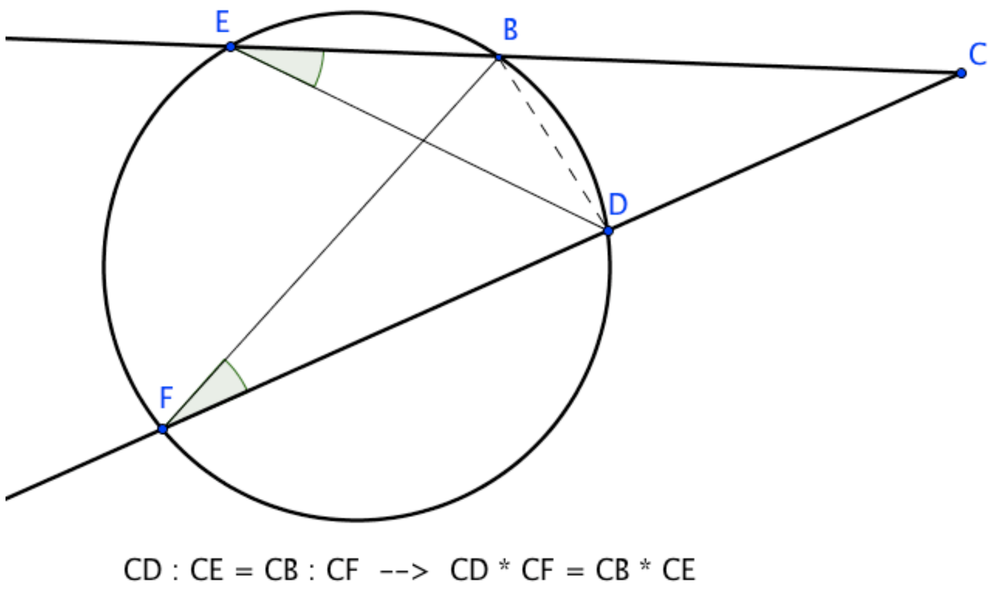
Polar equation: $r^4 + a^4 - 2r^2 a^2 \cos(2\theta) = b^4$

A parametrization for Cassini's oval is $r(t) \cdot (\cos(t), \sin(t))$,

$$r^2(t) := a^2 \cos(2t) + \sqrt{(-a^4 + b^4) + a^4(\cos(2t))^2},$$

$t \in (0, 2\pi]$, and $a < b$. This parametrization only generates parts of the curve when $a > b$.

By default 3D-XplorMath shows how the product definition of the Cassinian ovals leads to a *ruler and circle* construction based on the following circle theorem about products of segments:



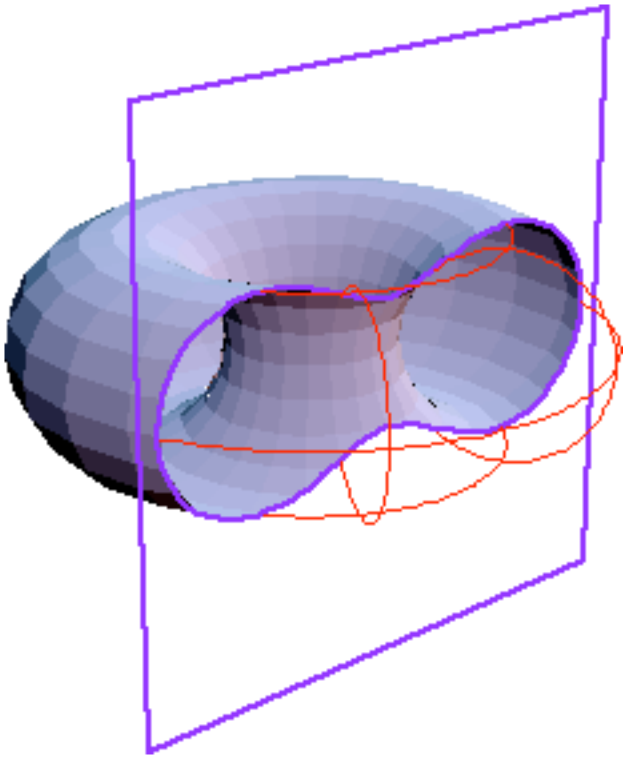
Cassinian Ovals as sections of a Torus

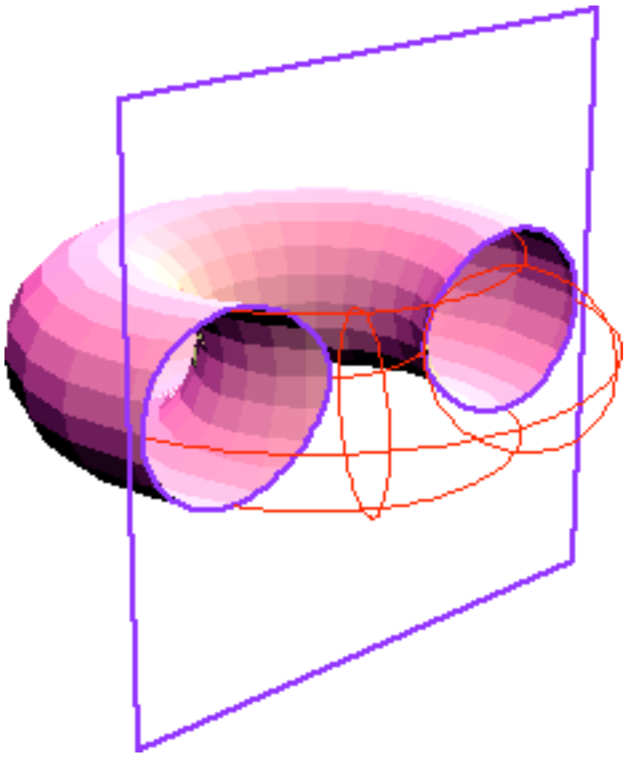
Let c be the radius of the generating circle and d the distance from the center of the tube to the directrix of the torus. The intersection of a plane c distant from the torus' directrix is a Cassinian oval, with $a = d$ and $b^2 = \sqrt{4}cd$, where a is half of the distance between foci, and b^2 is the constant product of distances.

Cassinian ovals with a large value of b^2 approch a circle, and the corresponding torus is one such that the tube radius is larger than the center to directrix, that is, a self-intersecting torus without the hole. This surface also approaches a sphere.

Note that the two tori in the figure below are not identical.

Arbitrary vertical slices of a torus are called Spiric Sections. In general they are *not* Cassinian ovals.





Proof: Start with the equation of a torus

$$(\sqrt{x^2+y^2}-d)^2+z^2=c^2.$$

Insert $y=c$, rearrange and square again:

$$x^2+z^2+d^2=2d\sqrt{x^2+c^2},\ (x^2+z^2+d^2)^2=4d^2(x^2+c^2).$$

Now multiply the factors of the implicit equation of an Cassinian oval and rearrange

$$\begin{aligned}((x-a)^2+y^2)\cdot((x+a)^2+y^2)&=b^4,\\ (x^2-a^2)^2+y^4+2y^2(x^2+a^2)&=b^4,\\ (x^2+y^2)^2+2a^2(y^2-x^2)&=b^4-a^4.\end{aligned}$$

These two equations match because of $a=d,\ b^2=2dc$, after rotation of the y-axis into the z-axis.

Curves that are the locus of points the product of whose distances from n points is constant are discussed on pages 60–63 of Visual Complex Analysis by Tristan Needham.

XL.

User Defined Plane Curves in 3DXM*

Selection of one of these entries will open a dialog to enter the data the user wishes. Default examples are provided.

User Cartesian: enter $x(t) := \ldots,\ y(t) := \ldots$.

User Polar: enter $r(t) := \ldots,\ \varphi(t) := \ldots$.
The curve is $(r(t)\cos(\varphi(t)),\ r(t)\sin(\varphi(t)))$.

User Graph: enter $y(t) := \ldots$, implied is $x(t) := t$. The curve $(t, y(t))$ is the Graph of the function y. Three approximations are shown: *Taylor, Interpolation, Fourier.*

These are the explicitly parametrized user curves. The standard decorations are available: Parallel Curves, Generalized Cycoids, Osculating Circles, Family of Normals and their Envelope, Caustics from Rotated Normals.

User Implicit: enter level function $F(x, y) := \ldots$.
See the separate text: *Implicit Planar Curves* above, available also from the Documentation Menu (after selection of user defined implicit curve).

User Curvature: enter the curvature function $\kappa(s) := \ldots$.
The program assumes that the parameter s is arc length. See also the text below: *User Curves By Curvature*, again available from the Documentation Menu of 3DXM.

H.K.

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User Defined by Curvature*

A planar curve (parametrized by arc length) can be reconstructed from its curvature function $t \mapsto \kappa(t)$ as follows:

(1) take the antiderivative of κ, $\alpha(t) := \int^t \kappa(\sigma)d\sigma$,

(2) choose an initial point p, an initial tangent vector $\dot{c}(0)$ and an orthonormal basis $e_1 = \dot{c}(0),\ e_2$,

so the definition of curvature (namely $\kappa := |\ddot{c}|$, plus a sign convention) implies that,

(3) $\dot{c}(t) = e_1 \cdot \cos\alpha(t) + e_2 \cdot \sin\alpha(t)$.

Then one more integration,

(4) $c(t) = p + \int_0^t \dot{c}(\sigma)d\sigma$,

determines the curve. This description explains why the curvature is also called the "rotation speed" of the tangent vector field $\dot{c}(t)$.

In 3D-XplorMath one can select *User Curvature*. A dialog box opens and one can enter the desired curvature function. The initial point p is taken as the origin and the initial tangent is taken as the unit vector in the positive x-direction.

The parameter gg in this case defines a "precision divisor", that can be between 1 and 30. The size of the

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subintervals used in approximating the above integrals is $\delta := (tMax - tMin)/(tResolution - 1)$ if $gg = 1$, and in general it is δ/gg. If the curvature function κ becomes very large somewhere, and in particular if it is infinite at an endpoint of the interval $[tMin, tMax]$, it is a good idea to use a fairly large value of gg to counteract the resulting numerical inaccuracies that will occur in the evaluation of the integrals.

Note that 3D-XplorMath offers the same Action Menu Entries as for explicitly parametrized curves. For example try the caustics.

R.S.P.