Conic Sections and the Dandelin Spheres

See also (in the Planar Curves Category): Parabola, Ellipse, Hyperbola, Conic Section and their ATOs.

(1) Parabolae, Ellipses and Hyperbolae have a three-dimensional definition as planar sections of (usually circular) cones, and they have a two-dimensional definition as loci having equal distance from a point and a line resp. as loci having the sum (or the difference) from two points being constant. The Dandelin Spheres explain why the different definitions give the same curves.

The program illustration shows:

(i) A cone and two inscribed "Dandelin Spheres". The two spheres are tangent to the cone along highlighted circles C_1, C_2.

(ii) A plane intersecting the cone in a highlighted curve E and also tangentially intersecting the two spheres in highlighted points F_1, F_2.

The illustration is meant to explain why it is that for each point P on the intersection curve E the sum of its distances to F_1, F_2 is constant, namely it is equal to the distance between the circles C_1, C_2 on the cone. To see this, note that all tangent segments from P to each of the spheres have the **same** length, so that the distances from P to F_j and from P to $C_j,\ j = 1,2$ are equal. This shows that E is an ellipse:

$$|P - F_1| + |P - F_2| = \text{distance}(C_1, C_2).$$

(2) In the Planar Curve ATO on Conic Sections we have introduced the directrix of a conic section as the line such that the **ratio** of the distances from a point P on the conic to the Focus F_j and to the directrix $D_j,\ j = 1,2$ is constant ($= 1$ for Parabolae, < 1 for Ellipses and > 1 for Hyperbolae). In the program illustration these directrices are highlighted, they are the two intersection lines of the plane of E with the two parallel planes of the circles C_1, C_2. We said already that the distances from P to F_j and from P to $C_j,\ j = 1,2$ are equal for all P. We therefore have to prove that for all P on the ellipse the ratio

$$\text{distance}(P, D_j) : \text{distance}(P, C_j).$$

is the same, namely it is equal to the ratio

$$\text{distance}(D_1, D_2) : \text{distance}(C_1, C_2).$$

But this follows since the two triangles made out of the segments $\overline{PD_j}$ and $\overline{PC_j}$ (and closed by a segment of D_j) for $j = 1$ and $j = 2$ are similar.

H.K.