The Clifford Tori *

The real Clifford tori are embeddings of the torus into the unit sphere $\mathbb{S}^3$ of $\mathbb{R}^4$, by $(u, v) \to Q(u, v) := (w, x, y, z)$, where

$$w = \cos(aa) \cos(u)$$

$$x = \cos(aa) \sin(u)$$
$$y = \sin(aa) \cos(v)$$
$$z = \sin(aa) \sin(v)$$

(Note that this is just the product of a circle in the (w, x) plane with a second circle in the (y, z) plane, and so is clearly flat.) To get something that we can see in $\mathbb{R}^3$, we stereographically project $\mathbb{S}^3$; i.e., the Clifford tori in $\mathbb{R}^3$ are the embeddings $(u, v) \to P(Q(u, v))$ where, $P : \mathbb{S}^3 \to \mathbb{R}^3$ is stereographic projection.

We take as the center of the stereographic projection map the point $(\cos(cc\,\pi), 0, \sin(cc\,\pi), 0)$. Varying cc deforms a torus of revolution through cyclides (presently in the Hopf-fibred case only).

The Clifford tori (in $\mathbb{S}^3$) are fibered by the Hopf fibers, and we show two versions of the stereographically projected Clifford tori, one parameterized by curvature lines and the other by Hopf fibers.

*This file is from the 3D-XploreMath project.
Please see http://vmm.math.uci.edu/3D-XplorMath/index.html

(To get the explicit parametrization of the latter, in the above formulae, replace u by $u + v$ and v by $u - v$.)

What is classically called *the* Clifford torus corresponds to $aa = \pi/4$. It has maximal area among the family and divides $\mathbb{S}^3$ into two parts of equal volume, but the other leaves of the foliation obtained by varying aa are also interesting. These tori are special cases of the flat Pinkall Tori in $\mathbb{S}^3$, and are discussed in more detail in its "About This Object...".

By morphing $0 \leq ff \leq 2\,\pi$, we rotate the torus in $\mathbb{S}^3$ around the Hopf fibre $v = 0$. One member of this family passes through the center of the stereographic projection, and its image in $\mathbb{R}^3$ is a once-punctured torus with a planar end. One observes in $\mathbb{R}^3$ a conformal deformation that, by allowing to pass through infinity, turns the torus inside out. The 180 degree rotation in this family is a conformal anti-involution of the torus which has the Hopf fibre $v = 0$ as its connected fixed point set. If one tries to observe such a deformation, the eye gets deceived and sees a slightly deformed rotation, and we therefore recommend to view it using the default two-sided user coloration.