Curvature Properties of Surfaces *

Any curvature discussion of surfaces assumes some knowledge about curvature properties of curves.

Planar Curves have, at each point $c(s)$, only one kind of curvature. Consider a circle through $c(s)$ that has the same first and second derivative as the curve at $c(s)$. Such a circle is called *osculating circle*, it approximates the curve better than any other circle and it can easily be recognized if $c'''(s) \neq 0$: The circle has the same tangent as the curve and is on different sides of the curve before and after $c(s)$. See `Osculating Circles` in the Action Menu. The radius of this circle is called *curvature radius* $r(s)$, and the *curvature* is defined as $\kappa(s) := 1/r(s)$. If s is the arclength parameter, i.e. $|c'(s)| = 1$, then $\kappa(s) = |c''(s)|$. The *fundamental theorem* for planar curves states:

> If a continuous curvature function $\kappa(s)$ is given then there exists, up to congruence, exactly one planar curve with this curvature function.

Space Curves have the same definition of *osculating circle* and of *curvature* $\kappa(s)$ as the plane curves. If s is arclength on c then $c''(s)$ is called *principal curvature vector* and $\vec{h}(s) = c''(s)/|c''(s)|$ is called *principal normal* .

In addition, space curves have a second kind of curvature, the *rotation speed* of the principal normal, also called *torsion* $\tau(s) := |\vec{h}'(s)|$. The *fundamental theorem* for space

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/

curves (roughly) states:

> If continuous curvature functions $\kappa(s), \tau(s)$ are given then there exists, up to congruence, exactly one space curve with these curvature functions.

Surfaces have as their most visible curvature properties their *normal curvatures*: Consider at a surface point p the intersection of the surface with all its normal planes at p. The curvatures of these *normal sections* are the normal curvatures. They can be computed as follows: Let N be a unit normal field along the surface and $c(s), c(0) = p$ a normal section with $c'(0) = \vec{e}, |\vec{e}| = 1$. Then its curvature, the normal curvature in the direction $\vec{e}$, is

$$\kappa(p, \vec{e}) = |c''(0)| = \langle D_{\vec{e}} N, \vec{e} \rangle.$$

These normal curvatures have a minimum and a maximum, called the principal curvatures κ_1, κ_2 at p. The corresponding vectors $\vec{e}$ are called the *principal directions* $\vec{e}_1 \perp \vec{e}_2$. $H := \kappa_1 + \kappa_2$ and $K := \kappa_1 \cdot \kappa_2$ are *mean curvature* and *Gauss curvature*.
If the surface is given by an *explicit parametrization*, it is straight forward to compute these data. If the surface is given by an *implicit equation* $f(x, y, z) = 0$, one chooses $N(x, y, z) := \operatorname{grad} f / |\operatorname{grad} f|$ and computes $\langle D_{\vec{e}} N, \vec{e} \rangle$, as before.
For these surfaces one finds in the Action Menu the entries:
`Add Principal Curvature Fields`,
`Move Principal Curvature Circles`.
They allow to view and move the above curvature objects.